Chapter 8

Combining Functions

Throughout this course, you have learned advanced techniques for interpreting a variety of functions. Understanding functional relationships between variables is a cornerstone to further study at the university level in disciplines such as engineering, physical sciences, business, and social sciences.

Relationships between variables can become increasingly complex and may involve a combination of two or more functions. In this chapter, you will learn techniques to analyse various combinations of functions and solve real-world problems requiring these techniques.

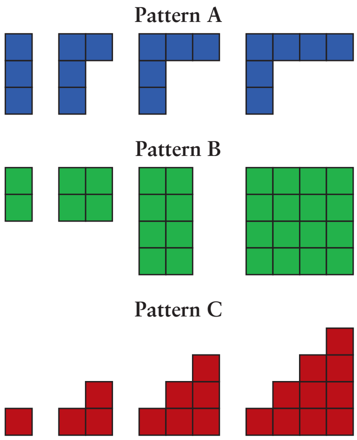
By the end of this chapter, you will

- determine, through investigation using graphing technology, key features of the graphs of functions created by adding, subtracting, multiplying, or dividing functions, and describe factors that affect these properties (D2.1)
- recognize real-world applications of combinations of functions, and solve related problems graphically (D2.2)
- determine, through investigation, and explain some properties of functions formed by adding, subtracting, multiplying, and dividing general functions (D2.3)
- determine the composition of two functions numerically and graphically, with technology, for functions represented in a variety of ways, and interpret the composition of two functions in real-world applications (D2.4)
- determine algebraically the composition of two functions, verify that $f(g(x))$ is not always equal to $g(f(x))$, and state the domain and the range of the composition of two functions (D2.5)
- solve problems involving the composition of two functions, including problems arising from real-world applications (D2.6)
- demonstrate, by giving examples for functions represented in a variety of ways, the property that the composition of a function and its inverse function maps a number onto itself (D2.7)
- make connections, through investigation using technology, between transformations of simple functions $f(x)$ and the composition of these functions with a linear function of the form $g(x) = A(x + B)$ (D2.8)
- compare, through investigation using a variety of tools and strategies, the characteristics of various functions (D3.1)
- solve graphically and numerically equations and inequalities whose solutions are not accessible by standard algebraic techniques (D3.2)
- solve problems, using a variety of tools and strategies, including problems arising from real-world applications, by reasoning with functions and by applying concepts and procedures involving functions (D3.3)

Prerequisite Skills

Identify Linear, Quadratic, and Exponential Growth Models

Use the three patterns shown to answer questions 1 to 3.



1. a) Build or draw the next stage in each pattern.

b) Identify the nature of growth for each pattern as linear, quadratic, exponential, or other. Justify your reasoning in each case.

2. a) Create a scatter plot of the total number, C, of tiles as a function of the stage number, n, for each growing pattern.

b) Do the shapes of the graphs confirm your answers to question 1b)? Explain why or why not.

3. Determine an equation relating C and n for each pattern in question 1. Verify that each equation holds true for the given pattern.

Graph and Analyse Power Functions

4. Sketch a graph of each power function and identify its domain and range.

a) $y = x$

b) $y = x^2$

c) $y = x^3$

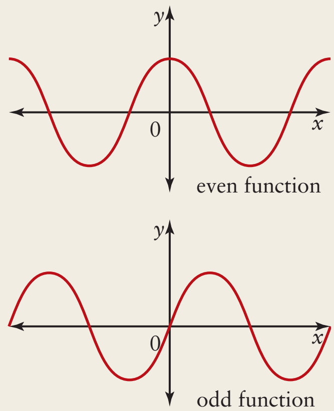
d) $y = x^4$

5. Identify whether each function in question 4 is even or odd.

CONNECTIONS

You studied power functions in Chapter 1.

Recall that an even function has line symmetry about the y-axis, and an odd function has point symmetry about the origin.



6. Use Technology Verify your answers to question 5 using graphing technology.

Graph and Analyse Rational Functions

7. Sketch a graph of each rational function and identify its domain and range.

a) $f(x) = \frac{1}{x}$

b) $g(x) = \frac{1}{x - 4}$

8. Simplify each rational function. State any restrictions on the variables.

a) $u(x) = \frac{x - 2}{x^2 - 4}$

b) $v(x) = \frac{x^2 - x - 6}{x + 2}$

9. Refer to question 8. For each function,

i) identify the domain and range

ii) graph the function

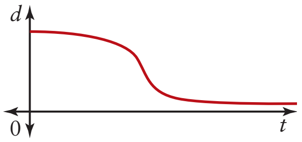
iii) identify any asymptotes or holes in the graph

10. Use Technology Verify your answers to question 9 using graphing technology.

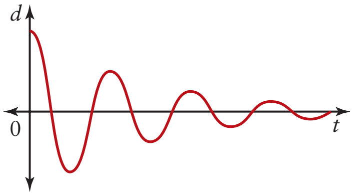
11. Match each distance-time graph with the scenario that it best represents. Give reasons for your answers.

Graph

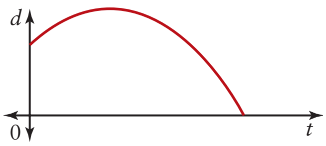
a)



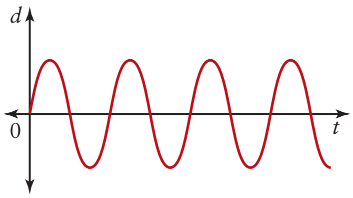
b)



c)



d)



Scenario

i) the displacement of the tip of a metronome

ii) a rocket's height after being launched from a raised platform

iii) a girl's displacement on a swing after receiving a push from her big brother

iv) the height of a glider drifting to the ground

12. Refer to question 11. Suggest a reasonable scale and unit of measure for each scenario. Explain your choices.

Inverses

13. Find the inverse of each function.

a) $f(x) = x - 2$

b) $g(x) = 4x + 3$

c) $h(x) = x^2 - 5$

d) $k(x) = \dfrac{1}{x + 1}$

14. Which inverses in question 13 are functions? Explain.

CHAPTER PROBLEM

As a recent graduate of the business program of an Ontario university, you have been hired by Funky Stuff, a company that manufactures games and toys for kids of all ages. As a member of the marketing department, you will be given a number of scenarios and asked to identify marketing and pricing strategies to maximize profits for the company. Your understanding of functions and how they can be combined will be very useful as you begin your business career.

8.1 Sums and Differences of Functions

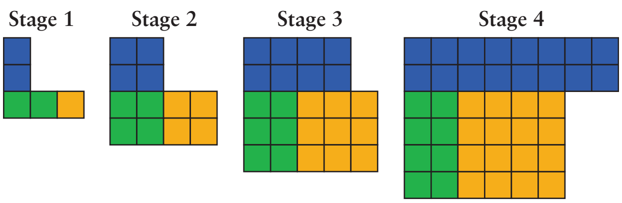
Have you ever swum in a wave pool? If you have, you may have noticed that in some spots the waves can get quite high, while in other spots there is very little wave motion. Why do you suppose this happens? A sinusoidal function can be used to model a single wave, but when there are two or more waves interfering, the mathematics involved can get a little deeper.

Functions can be combined in a variety of ways to describe all sorts of complicated relationships. In many cases, such as the interference of two sine waves, addition and subtraction can be applied.

Investigate How can you represent the addition of functions in various ways?

Tools
- coloured tiles or linking cubes
- graphing calculator or graphing software

Examine the pattern of tiles shown.



1. Copy the table. Complete the first four rows for the coloured tiles. Leave the last column blank for now.

Stage Number, n	Blue Tiles, Y_1	Green Tiles, Y_2	Yellow Tiles, Y_3	
1				
2				
3				
4				
⋮	⋮	⋮	⋮	⋮
n				

2. a) Predict the number of each coloured tile required to build the next stage in the sequence.

b) Build or draw the next stage.

3. Examine the patterns in each tile column.

a) Identify which colour of tile is growing

i) linearly

ii) quadratically

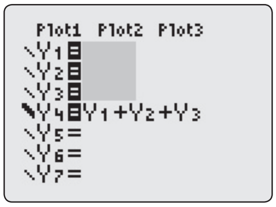
iii) exponentially

b) Write an equation to represent each coloured tile pattern in the last row of the table.

4. The **superposition principle** states that the sum of two or more functions can be found by adding the ordinates (y-coordinates) of the functions at each abscissa (x-coordinate).

Define Y_4 as the sum of Y_1, Y_2, and Y_3. Use the superposition principle to graph Y_4 on a graphing calculator.

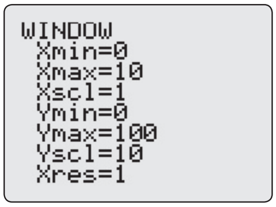
First, use the **Y=** editor to enter the three functions as **Y1**, **Y2**, and **Y3**, respectively.



Then, do the following for **Y4**:

- Press VARS and cursor over to the **Y-VARS** menu.
- Select **1:Function...** and then **1:Y1**.
- Press + and repeat to enter the other functions, as shown.
- Change the line style of **Y4** to heavy.

Use the window settings shown. Press GRAPH to view all four functions.



5. a) Determine the values of Y_4 when $x = 1$, $x = 2$, $x = 3$, and $x = 4$.

- Press 2nd [CALC] to display the **CALCULATE** menu, and select **1:value**.
- Enter one of the x-values and press ENTER.
- Use the up and down cursor keys to move from one graph to another, if necessary.

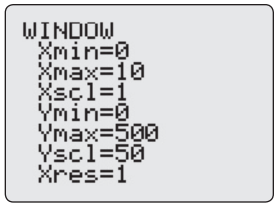
What meaning do these numbers have with respect to the growing pattern of tiles?

b) Complete the last column of the table. Use the heading "Total Number of Tiles, $Y_1 + Y_2 + Y_3$," and enter the values. Compare these values to those obtained in part a). Explain this result.

c) In the last row of the last column, write an equation to model the total number of tiles in any stage, n, of the pattern.

6. **Reflect**

a) View the graph representing the total tiles, Y_4, using the window settings shown.



Describe the shape of this graph. Explain why the graph looks the way it does.

b) Summarize the ways in which the total tiles pattern relationship is represented in this activity.

Example 1 Apply the Superposition Principle

Determine an equation for the function $h(x) = f(x) + g(x)$ in each case. Then, graph $h(x)$ and state the domain and range of the function.

a) $f(x) = x^2$, $g(x) = 3$

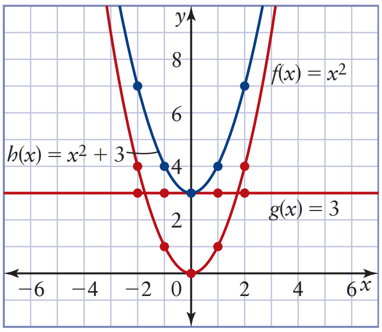
b) $f(x) = x^2$, $g(x) = x$

Solution

a) An equation for $h(x)$ can be found by adding the expressions for $f(x)$ and $g(x)$.

$$h(x) = f(x) + g(x)$$
$$= x^2 + 3$$

To produce the graph of $h(x)$ using the superposition principle, first graph $f(x)$ and $g(x)$ on the same set of axes. Then, graph the sum of these functions, $h(x)$, by adding the y-coordinates at each point.



x	$f(x) = x^2$	$g(x) = 3$	$h(x) = x^2 + 3$
−2	4	3	4 + 3 = 7
−1	1	3	1 + 3 = 4
0	0	3	0 + 3 = 3
1	1	3	1 + 3 = 4
2	4	3	4 + 3 = 7

Note that this superposition can be thought of as a *constant* vertical translation of the parabola $y = x^2$ to produce the parabola $y = x^2 + 3$.

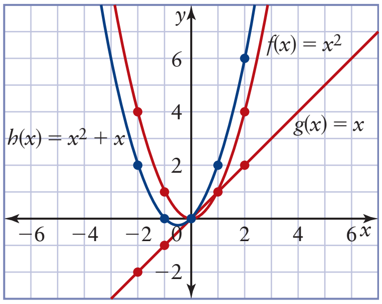
From the graph or the equation, it is clear that the domain of $h(x)$ is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}, y \geq 3\}$.

b) $h(x) = f(x) + g(x)$

$\quad\;\; = x^2 + x$

First, graph $f(x)$ and $g(x)$ on the same set of axes.

Then, graph the sum of these functions, $h(x)$, by adding the y-coordinates at each point.



x	$f(x) = x^2$	$g(x) = x$	$h(x) = x^2 + x$
−2	4	−2	4 + (− 2) = 2
−1	1	−1	1 + (−1) = 0
0	0	0	0 + 0 = 0
1	1	1	1 + 1 = 2
2	4	2	4 + 2 = 6

Note that this superposition can be thought of as a *variable* vertical translation of the parabola $y = x^2$ to produce the function $y = x^2 + x$.

The domain of $h(x)$ is $x \in \mathbb{R}$.

To identify the range, first find the vertex, which is the minimum point, since the parabola opens upward.

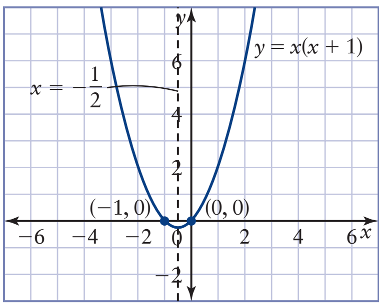
Method 1: *Apply Algebraic and Graphical Reasoning*

Locate the x-coordinate of the vertex, and then find the corresponding value of y at that point.

$y = x^2 + x$

$\;\; = x(x + 1)$ Write in factored form.

The zeros of this function occur when $x = 0$ and $x + 1 = 0$, or $x = -1$. The zeros are 0 and -1. The minimum value will occur midway between these x-coordinates, due to symmetry.



The minimum value occurs when $x = -\frac{1}{2}$. Evaluate y at this point.

$$
\begin{aligned}
y &= x^2 + x \\
&= \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) \\
&= \frac{1}{4} - \frac{1}{2} \\
&= -\frac{1}{4}
\end{aligned}
$$

The minimum value of y is $-\frac{1}{4}$. Therefore, the range of $h(x)$ is $\left\{y \in \mathbb{R}, y \geq -\frac{1}{4}\right\}$.

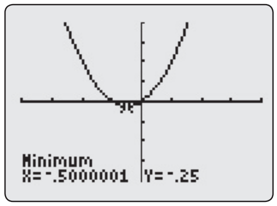
Method 2: *Use a Graphing Calculator*

Graph $h(x) = x^2 + x$. Use the **Minimum** operation.

- Press (2nd) [CALC] to display the **CALCULATE** menu, and then select **3:minimum**.
- Move the cursor to locations for the left bound, right bound, and guess, pressing (ENTER) after each.

  The minimum value of y is $-\frac{1}{4}$.

  Therefore, the range of $h(x)$ is $\left\{y \in \mathbb{R}, y \geq -\frac{1}{4}\right\}$.



The superposition principle can be extended to the difference of two functions, because subtracting is the same as adding the opposite.

Example 2 The Profit Function

Student Council is selling T-shirts to raise money for new volleyball equipment. There is a fixed cost of $200 for producing the T-shirts, plus a variable cost of $5 per T-shirt made. Council has decided to sell the T-shirts for $8 each.

a) Write an equation to represent
- the total cost, C, as a function of the number, n, of T-shirts produced
- the revenue, R, as a function of the number, n, of T-shirts produced

Then, graph these functions on the same set of axes. Identify the point of intersection and explain the meaning of its coordinates.

b) Profit, P, is the difference between revenue and expenses. Develop an algebraic and a graphical model for the profit function.

c) Under what circumstances will Student Council lose money? make money?

d) Identify the domain and range of the cost, revenue, and profit functions in the context of this problem.

Solution

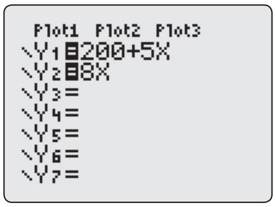
a) The total cost of producing the T-shirts is the sum of the fixed cost and the variable cost:

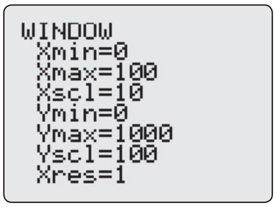
$C(n) = 200 + 5n$

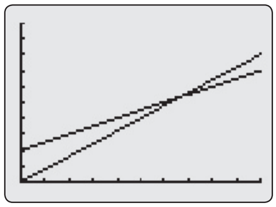
The revenue is \$8 per T-shirt multiplied by the number of T-shirts sold:

$R(n) = 8n$

Use a graphing calculator to graph these functions on the same set of axes. Use number sense to choose appropriate window settings.







Determine the point of intersection.

Method 1: *Use Pencil and Paper*

To find the point of intersection, solve the following linear system.

$C(n) = 200 + 5n$
$R(n) = 8n$

The expressions on the right side above will have the same value when $R = C$.

$$8n = 200 + 5n$$
$$3n = 200$$
$$n = \frac{200}{3}$$

Find the corresponding value of R.

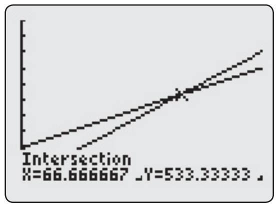
$$R = 8n$$ Substitute into either equation. Pick the easier one.
$$= 8\left(\frac{200}{3}\right)$$
$$= \frac{1600}{3}$$

The point of intersection is $\left(\frac{200}{3}, \frac{1600}{3}\right)$, or $\left(66\frac{2}{3}, 533\frac{1}{3}\right)$.

Method 2: *Use a Graphing Calculator*

Use the **Intersect** operation.

- Press [2nd] [CALC] to display the **CALCULATE** menu, and then select **5:intersect**.
- Use the cursor keys to select the first relation, the second relation, and guess, pressing [ENTER] after each.



The point of intersection is $(66.\overline{6}, 533.\overline{3})$, or $\left(66\frac{2}{3}, 533\frac{1}{3}\right)$.

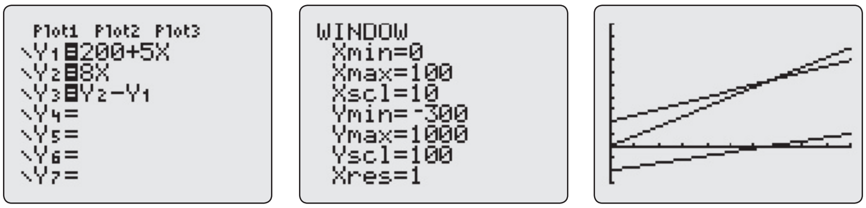
This means that if Student Council sells approximately 67 T-shirts, then the revenue and the cost will be equal, at approximately \$533. This is known as the **break-even point**.

b) Profit is the difference between revenue and expenses.

$$\begin{aligned} P(n) &= R(n) - C(n) \\ &= 8n - (200 + 5n) \\ &= 8n - 200 - 5n \\ &= 3n - 200 \end{aligned}$$

This equation represents the profit as a function of the number of T-shirts sold.

Use the superposition principle to develop a graphical model. Adjust the window settings to view the domain of interest.



c) Student Council will lose money if profit is less than zero. This is modelled by the section of the graph of $P(n)$ that appears below the horizontal axis. Student Council will make money when $P(n) > 0$, or if more than 67 T-shirts are sold.

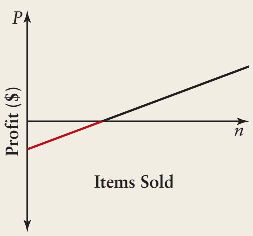
d) Although each linear function extends to the left of the vertical axis, it has no meaning for $n < 0$, because it is impossible to sell a negative number of T-shirts. Inspection of the graphs reveals the domain and range for these functions in the context of this problem.

Function	Practical Domain	Practical Range
$C(n) = 200 + 5n$	$\{n \in \mathbb{Z}, n \geq 0\}$	$\{C \in \mathbb{Z}, C \geq 200\}$
$R(n) = 8n$	$\{n \in \mathbb{Z}, n \geq 0\}$	$\{R \in \mathbb{Z}, R \geq 0\}$
$P(n) = 3n - 200$	$\{n \in \mathbb{Z}, n \geq 0\}$	$\{P \in \mathbb{Z}, P \geq -200\}$

Note that n is restricted to integer values greater than or equal to zero, while each of the dependent variables has different restrictions.

CONNECTIONS

In business, *in the red* means *absorbing a loss* and *in the black* means *making a profit*.



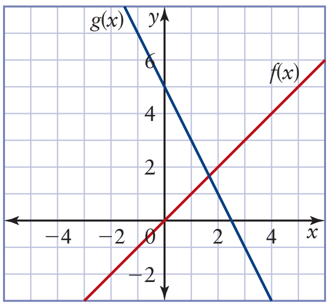
These phrases reflect the historical accounting practice of using red ink to write a loss and black ink to write a profit.

KEY CONCEPTS

- Some combined functions are formed by adding or subtracting two or more functions.
- The superposition principle states that the sum of two functions can be found by adding the y-coordinates at each point along the x-axis.
- The superposition principle also applies to the difference of two functions.
- The domain of the sum or difference of functions is the domain common to the component functions.

Communicate Your Understanding

C1 **a)** Explain how you can produce the graph of the combined function $y = f(x) + g(x)$ from the graph shown, where $f(x) = x$ and $g(x) = -2x + 5$.



b) Explain how you can produce the graph of the combined function $y = f(x) - g(x)$.

C2 Refer to Example 1, part b). Explain what is meant by the term *variable vertical translation* in this situation.

C3 Refer to Example 2.

a) Why is it necessary to restrict the domain of n to integers greater than or equal to zero?

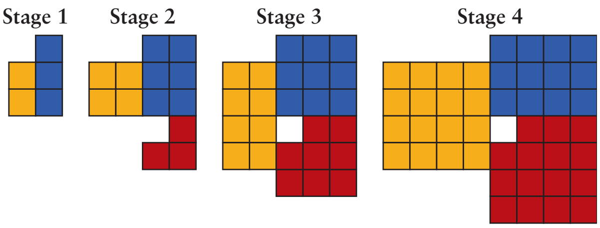
b) Why is this not necessary for the dependent variables, C and P?

C4 When dealing with a cost-revenue system of equations, what is meant by the break-even point? Why is this point important? Illustrate your answer with a sketch.

A Practise

For help with questions 1 and 2, refer to the Investigate.

Use the pattern of tiles shown to answer questions 1 and 2.



1. a) Identify the part of the pattern that is growing

i) linearly

ii) quadratically

iii) exponentially

b) Develop an equation for each of these parts, and verify that each equation holds true.

2. a) Develop an equation that gives the total number of tiles for each stage.

b) Use this equation to predict the total number of tiles required for

i) the fifth stage

ii) the sixth stage

c) Check your prediction by extending the model using concrete materials or a diagram.

For help with questions 3 to 9, refer to Examples 1 and 2.

3. For each pair of functions, find

i) $y = f(x) + g(x)$

ii) $y = f(x) - g(x)$

iii) $y = g(x) - f(x)$

a) $f(x) = 5x$ and $g(x) = x + 7$

b) $f(x) = -2x + 5$ and $g(x) = -x + 9$

c) $f(x) = x^2 + 4$ and $g(x) = 1$

d) $f(x) = -3x^2 + 4x$ and $g(x) = 3x - 7$

4. Let $f(x) = 4x + 3$ and $g(x) = 3x - 2$.

a) Determine an equation for the function $h(x) = f(x) + g(x)$. Then, find $h(2)$.

b) Determine an equation for the function $j(x) = f(x) - g(x)$. Then, find $j(-1)$.

c) Determine an equation for the function $k(x) = g(x) - f(x)$. Then, find $k(0)$.

5. Let $f(x) = -4x^2 + 5$ and $g(x) = 2x - 3$.

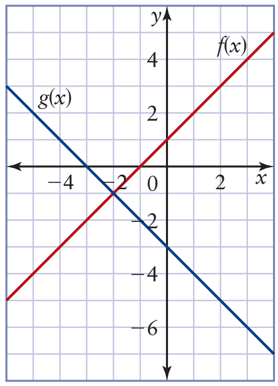
a) Determine an equation for the function $h(x) = f(x) + g(x)$. Then, find $h(-3)$.

b) Determine an equation for the function $j(x) = f(x) - g(x)$. Then, find $j(0)$.

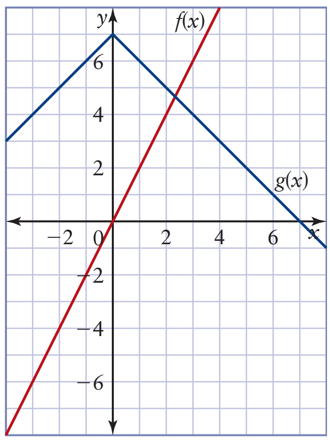
c) Determine an equation for the function $k(x) = g(x) - f(x)$. Then, find $k(3)$.

6. Copy each graph of $f(x)$ and $g(x)$. Then, apply the superposition principle to graph $f(x) + g(x)$. Give the domain and range of $f(x) + g(x)$.

a)



b)



7. For each graph in question 6, use the superposition principle to graph $y = g(x) - f(x)$.

8. Let $f(x) = 2^x$ and $g(x) = 3$.

a) Graph each of the following.

i) $y = f(x) + g(x)$

ii) $y = f(x) - g(x)$

iii) $y = g(x) - f(x)$

b) Explain how you could also produce each of these combined functions by applying transformations to the graph of $f(x) = 2^x$.

c) Give the domain and the range of each combined function.

9. Let $f(x) = \sin x$ and $g(x) = \log x$. Work in radians.

a) Sketch these functions on the same set of axes.

b) Sketch a graph of $y = f(x) + g(x)$.

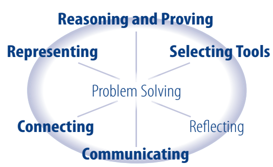
c) Sketch a graph of $y = f(x) - g(x)$.

d) **Use Technology** Check your answers using graphing technology.

B Connect and Apply

For help with questions 10 and 11, refer to Example 2.

10. A hotdog vendor has fixed costs of \$120 per day to operate, plus a variable cost of \$1 per hotdog sold. He earns \$2.50 per hotdog sold, in revenue. The maximum number of hotdogs that he can sell in a day is 250.



a) Write an equation to represent

i) the total cost, C, as a function of the number, h, of hotdogs sold

ii) the revenue, R, as a function of the number, h, of hotdogs sold

b) Graph $C(h)$ and $R(h)$ on the same set of axes.

c) Identify the break-even point and explain what its coordinates mean.

d) Develop an algebraic and a graphical model for the profit function, $P(h) = R(h) - C(h)$.

e) Identify the domain and range in the context of this problem for $C(h)$, $R(h)$, and $P(h)$.

f) What is the maximum daily profit the vendor can earn?

11. Refer to question 10. The hotdog vendor has found a way to improve the efficiency of his operation that will allow him to either reduce his fixed cost to \$100 per day *or* reduce his variable cost to \$0.90 per hotdog.

a) Which of these two options has the most favourable effect on

i) the break-even point?

ii) the potential maximum daily profit?

b) What advice would you give the hotdog vendor?

12. An alternating current–direct current (AC-DC) voltage signal is made up of the following two components, each measured in volts (V):

$V_{AC} = 10 \sin t$ $\qquad$ $V_{DC} = 15$

a) Sketch graphs of these functions on the same set of axes. Work in radians.

b) Graph the combined function $V_{AC} + V_{DC}$.

c) Identify the domain and range of $V_{AC} + V_{DC}$.

d) Use the range of the combined function to determine the following values of this voltage signal:

i) minimum

ii) maximum

iii) average

13. **Use Technology** Consider the combined function $T(x) = f(x) + g(x) + h(x)$, where $f(x) = 2^x$, $g(x) = 2x$, and $h(x) = x^2$.

a) How is this function related to the total tiles pattern in the Investigate at the beginning of this section?

b) Graph $f(x)$, $g(x)$, and $h(x)$ on the same set of axes. Use colours or different line styles to easily distinguish the curves.

c) Graph the combined function $T(x)$.

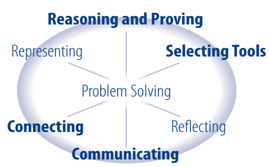
d) Adjust the viewing window to explore what happens as the x- and y-values are increased by large quantities. The function $T(x)$ appears to converge with one of the three component functions. Which one is it?

e) Explain the result in part d) by considering the rates of change of the component functions.

14. **a)** Is $f(x) + g(x) = g(x) + f(x)$ true for all functions $f(x)$ and $g(x)$? Use examples to support your answer.

b) Is $f(x) - g(x) = g(x) - f(x)$ true for all functions $f(x)$ and $g(x)$? Use examples to support your answer.

c) What can you conclude about the commutative property of the sum of two functions? Does it hold true? What about the difference of two functions?



CONNECTIONS

When the order of operands can be reversed and still produce the same result, then the operation is said to be commutative. For example,

$3 + 5 = 5 + 3$

$a + b = b + a$ for any $a, b \in \mathbb{R}$.

Addition of real numbers is commutative.

15. **Use Technology**

a) Graph the function $f(x) = \sin x$ using *The Geometer's Sketchpad*® or a graphing calculator. Work in radians.

b) If you are using *The Geometer's Sketchpad*®, create a parameter c and set its initial value to zero. If you are using a graphing calculator, store the value zero in variable C.

c) Graph the function $g(x) = \sin(x - c)$. Explain how $f(x)$ and $g(x)$ are related when $c = 0$.

d) Predict the shape of $h(x) = f(x) + g(x)$. Sketch a graph of your prediction. Graph the function $h(x)$ and check your prediction. Use line colouring and/or bold to distinguish the graphs.

e) Give the domain and range of $h(x)$.

f) Predict the shape of $h(x)$ when the value of c is changed to each of these values:

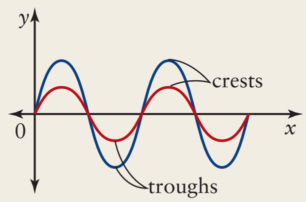
i) $\frac{\pi}{2}$ **ii)** π **iii)** 2π

g) Adjust the value of c to check your predictions. Give the domain and range of $h(x)$ in each case. If you are using *The Geometer's Sketchpad*®, you can use the motion controller to view the effects of continuously varying the value of c.

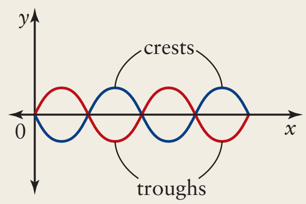
h) Explain how these results may help explain the wave action of wave pools.

CONNECTIONS

Two sinusoidal functions of the same frequency are *in phase* when their crests and troughs occur at the same points. *Constructive interference* occurs when the amplitude of the superposition is greater than the amplitude of each of the individual waves. This happens when two sinusoidal waves are in, or nearly in, phase.



If their crests and troughs occur at different points, the functions are *out of phase*. *Destructive interference* occurs when the amplitude of the superposition is less than the amplitude of each of the individual waves. Total destructive interference happens when two sinusoidal waves with the same frequency are out of phase by half a cycle.



These concepts are important in certain areas of physics, such as electronics and the theory behind electrical power generators.

16. Let $f(x) = x + 5$ and $g(x) = x + 2$.

a) Write an expression for $-g(x)$.

b) Graph $f(x)$ and $-g(x)$ on the same set of axes.

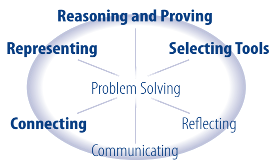
c) Add these functions to produce $f(x) + [-g(x)]$.

d) Graph $f(x)$ and $g(x)$ together on another set of axes.

e) Subtract these functions to produce $f(x) - g(x)$. Compare this result to the one obtained in part c).

f) Explain how this illustrates that the superposition principle works for subtracting functions.

17. Refer to question 16. Develop an algebraic argument that illustrates the same concept.



18. Chapter Problem The annual operating costs for Funky Stuff's Game Division are summarized in the table.

Operating Costs	
Fixed Costs	Variable Costs (per game)
Rent: \$12 000	Material: \$6
Taxes: \$3 000	Labour: \$9
Utilities: \$5 000	
TOTAL:	TOTAL:

a) What are the total fixed costs? Graph this as a function of the number of games, and explain the shape of the graph. Call this function Y_1.

b) What are the total variable costs? Graph this as a function of the number of games, and explain the shape of the graph. Call this function Y_2.

c) Use the superposition principle to graph $Y_3 = Y_1 + Y_2$. Explain what this function represents.

d) The revenue earned from sales is \$20 per game. Graph this as a function of the number of games, and explain the shape of the graph. Call this function Y_4.

e) Graph Y_3 and Y_4 on the same set of axes. Identify the point of intersection and explain the meaning of its coordinates. Why is this called the break-even point?

f) Use the superposition principle to graph $Y_5 = Y_4 - Y_3$. What does this function represent?

g) Explain the significance of Y_5

i) to the left of the x-intercept

ii) at the x-intercept

iii) to the right of the x-intercept

h) Describe what you expect to happen to the position of the break-even point if the game price is

i) increased

ii) decreased

Support your reasoning with graphs. Discuss any assumptions you must make.

Achievement Check

19. Let $Y_1 = x^2$ and $Y_2 = 2$. Use graphs, numbers, and words, to determine

a) the function $Y_3 = Y_1 + Y_2$

b) the range of Y_3

C Extend and Challenge

20. Use Technology Let $f(x) = \tan x$ and $g(x) = x$.

a) Use graphing technology to graph $f(x)$ and $g(x)$ on the same set of axes. Work in radians.

b) Predict the shape of $h(x) = f(x) + g(x)$. Sketch a graph of your prediction.

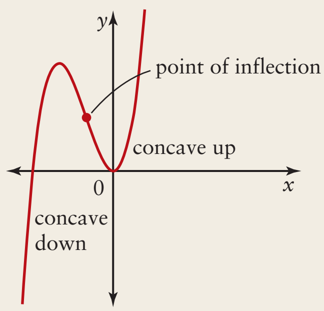
c) Use graphing technology to check your prediction.

d) Hide $f(x)$ so that only $g(x)$ and $h(x)$ are visible. How many intersection points do there appear to be? You may need to adjust the viewing window.

e) Describe what you notice about where the line $g(x) = x$ intersects the graph of $h(x)$. Consider the curvature (up versus down) of $h(x)$. Explain why this is so.

CONNECTIONS

A point on a graph where the curvature changes from concave up to concave down, or vice versa, is called a *point of inflection*.



You will learn more about points of inflection and their significance if you study calculus.

21. Refer to question 20. Repeat the analysis for $g(x) = -x$.

22. Can the superposition principle be extended to the multiplication or division of two functions? Carry out an investigation using examples of your choice. Summarize your findings.

23. Use an example to help you generate a hypothesis about whether the sum of two even functions is even, odd, or neither. Verify algebraically with a second pair of functions. (Use at least three different types of functions, drawn from polynomial, trigonometric, exponential/logarithmic, and rational). Illustrate your results with graphs.

24. Math Contest Given a polynomial $f(x)$ of degree 2, where $f(x + 1) - f(x) = 6x - 8$ and $f(1) = 26$, then $f(2)$ is

A 23

B 24

C 25

D 26

E none of the above

25. Math Contest For which value of a does $\frac{2x + 5}{3x + 8} = a$ not have a solution for x?

A $-\frac{5}{8}$

B $\frac{5}{8}$

C $\frac{8}{5}$

D $\frac{2}{3}$

E none of the above

26. Math Contest If $x = \frac{1}{4 - y}$, evaluate $\frac{1}{x} + 4x + y - yx - 1$.

27. Math Contest If the roots of $Ax^2 + Bx + C = 0$ are negative reciprocals of each other, determine the relationship between A and C. What can you say about A and C if the roots are simply reciprocals of each other?

8.2 Products and Quotients of Functions

The superposition principle is a convenient way to visualize the sum or difference of two or more functions. When two functions are combined through multiplication or division, however, the nature of the resultant combined function is less obvious. In earlier chapters and courses, you learned various techniques for multiplying and dividing algebraic expressions. Many of these skills can be applied and extended to products and quotients of functions.

Products and quotients of functions can be combined to model and solve a variety of problems. How are these concepts related to the revenue generated at baseball games?

Investigate What is the symmetrical behaviour of products and quotients of symmetrical functions?

Tools

- grid paper

A: *Products of Functions*

1. a) Sketch a graph of each power function.

i) $f(x) = x$ **ii)** $g(x) = x^2$ **iii)** $p(x) = x^3$ **iv)** $q(x) = x^4$

b) Classify each function as even, odd, or neither.

2. Use the functions from step 1. Prove each identity using algebraic reasoning.

a) $f(x)g(x) = p(x)$ **b)** $f(x)p(x) = q(x)$

c) $f(x)f(x) = g(x)$ **d)** $[g(x)]^2 = q(x)$

3. Examine the pattern of symmetry of the functions on the left side and right side of each identity in step 2. Copy and complete the table.

Identity	Symmetry of Factor Functions	Symmetry of Product Function
a) $f(x)g(x) = p(x)$	$f(x) = x$ (ODD) $g(x) = x^2$ (EVEN)	$p(x) = x^3$ (ODD)
b) $f(x)p(x) = q(x)$		
c) $f(x)f(x) = g(x)$		
d) $[g(x)]^2 = q(x)$		
Create an example of your own using power functions.		
Create an example of your own using power functions.		

4. Look for a pattern in the table. Write a conjecture about the nature of the symmetry of a combined function that consists of the product of
   a) two even functions
   b) two odd functions
   c) an even function and an odd function

5. **Reflect**
   a) Test your conjecture by exploring the graphs of various products of the following functions:

   $f(x) = \sin x$ $\quad g(x) = x$ $\quad h(x) = \cos x$

   $u(x) = x^2$ $\quad v(x) = \tan x$ $\quad w(x) = x^3$

   b) Based on your findings, write a summary of the symmetrical behaviour of the products of functions.

B: *Quotients of Functions*

1. Design and carry out an investigation to determine the symmetric behaviour of combined functions formed by the quotient of two symmetric functions. Write a brief report of your findings that includes
   - a comparison to the symmetrical behaviour of products of symmetric functions
   - the conditions under which asymptotic behaviour occurs

To simplify a combined function formed by the product or quotient of functions, some previously learned algebraic skills can be useful.

Example 1 Product and Quotient of Functions

Let $f(x) = x + 3$ and $g(x) = x^2 + 8x + 15$. Determine an equation for each combined function. Sketch a graph of the combined function and state its domain and range.

a) $y = f(x)g(x)$

b) $y = \dfrac{f(x)}{g(x)}$

Solution

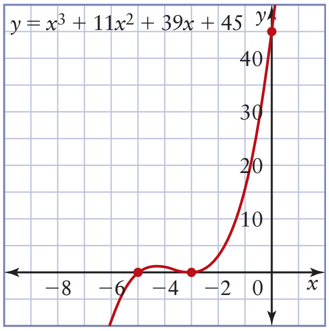
a) To determine $y = f(x)g(x)$, multiply the two functions.

$$\begin{aligned} y &= f(x)g(x) \\ &= (x + 3)(x^2 + 8x + 15) \\ &= x^3 + 8x^2 + 15x + 3x^2 + 24x + 45 && \text{Apply the distributive property.} \\ &= x^3 + 11x^2 + 39x + 45 && \text{Collect like terms.} \end{aligned}$$

Use the fact that $g(x)$ can be factored to help you sketch a graph of this cubic function.

$$y = (x + 3)(x^2 + 8x + 15)$$
$$= (x + 3)(x + 3)(x + 5)$$
$$= (x + 3)^2(x + 5)$$

The zeros of $y = (x + 3)^2(x + 5)$ are -3 (order 2) and -5. These are the x-intercepts. From the expanded form, $y = x^3 + 11x^2 + 39x + 45$, the y-intercept is 45. The leading coefficient is positive.



CONNECTIONS

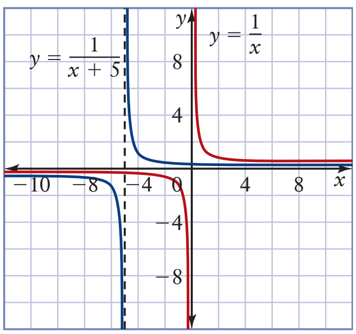
You explored polynomial functions in depth in Chapter 1.

There are no restrictions on x or y. The domain is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.

b) To determine $y = \frac{f(x)}{g(x)}$, divide the two functions.

$$y = \frac{f(x)}{g(x)}$$
$$= \frac{x + 3}{x^2 + 8x + 15}$$
$$= \frac{\cancel{x + 3}}{\cancel{(x + 3)}(x + 5)} \quad \text{Factor the denominator and simplify.}$$
$$= \frac{1}{x + 5}, x \neq -3$$

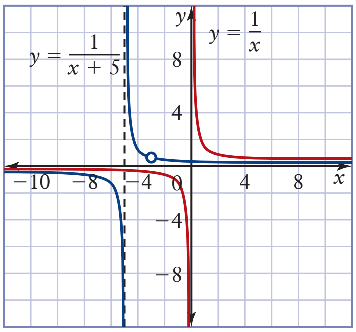
This is a rational function. A sketch of the graph can be produced by applying a horizontal translation of 5 units to the left of the function $y = \frac{1}{x}$.



To identify the domain of this function, refer to the original form of the function, not the simplified one:

$$\frac{f(x)}{g(x)} = \frac{x + 3}{(x + 3)(x + 5)} \qquad x + 3 \neq 0 \text{ and } x + 5 \neq 0.$$

Note that this function is undefined for $x = -3$ and $x = -5$. Therefore, the domain is $\{x \in \mathbb{R}, x \neq -3, x \neq -5\}$. This is partly reflected in the graph by the asymptote that appears at $x = -5$. However, the graph of this function must be adjusted to reflect the discontinuity at $x = -3$. There is a hole at $\left(-3, \frac{1}{2}\right)$.



There is a horizontal asymptote at $y = 0$, so the range is $\left\{y \in \mathbb{R}, y \neq 0, \frac{1}{2}\right\}$.

CONNECTIONS

You explored rational functions in depth in Chapter 3.

These algebraic and graphing skills are useful for solving problems involving products and quotients of functions.

Example 2 Baseball Marketing

In an effort to boost fan support, the owners of a baseball team have agreed to gradually reduce ticket prices, P, in dollars, according to the function $P(g) = 25 - 0.1g$, where g is the number of games that have been played so far this season.

The owners are also randomly giving away free baseball caps. The number, C, in hundreds, of caps given away per game can be modelled by the function $C(g) = 2 - 0.04g$.

Since these marketing initiatives began, the number, N, in hundreds, of fans in attendance has been modelled by the function $N(g) = 10 + 0.2g$.

a) Develop an algebraic and a graphical model for $f(g) = P(g)N(g)$ and explain what it means. Will the owners increase or decrease their revenue from ticket sales under their current marketing plan?

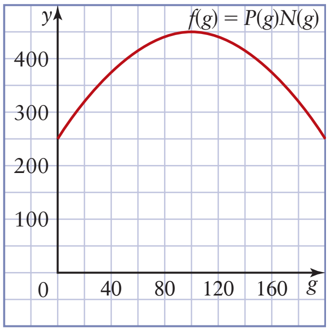
b) Develop an algebraic and a graphical model for $f(g) = \frac{C(g)}{N(g)}$ and explain what it means. How likely are you to receive a free baseball cap if you attend game 5?

Solution

a) Multiply $P(g)$ by $N(g)$ to produce the combined function $f(g) = P(g)N(g)$.

$$\begin{aligned} f(g) &= P(g)N(g) \\ &= (25 - 0.1g)(10 + 0.2g) \\ &= 250 + 5g - g - 0.02g^2 \\ &= -0.02g^2 + 4g + 250 \end{aligned}$$

A graph of this function is shown. Note that these functions only have meaning for $g \geq 0$.

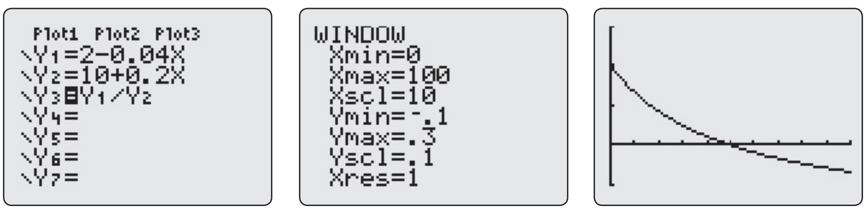


This combined function is the product of the ticket price and the number of fans attending, in hundreds. Therefore, $f(g) = P(g)N(g)$ represents the revenue from ticket sales, in hundreds of dollars. Note that the function is quadratic, increasing until about game 100, and then decreasing after that. Therefore, the owners will increase their revenue from ticket sales in the short term under their current marketing strategy, but eventually this strategy will no longer be effective.

b) Divide $C(g)$ by $N(g)$ to produce the combined function $f(g) = \dfrac{C(g)}{N(g)}$.

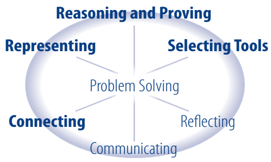
$$\begin{aligned} f(g) &= \frac{C(g)}{N(g)} \\ &= \frac{2 - 0.04g}{10 + 0.2g} \end{aligned}$$

Use a graphing calculator to graph this function. Enter the equations $C(g) = 2 - 0.04g$ and $N(g) = 10 + 0.2g$ first. Then, use them to produce the graph of $f(g) = \dfrac{C(g)}{N(g)}$. To view only the quotient function, leave it turned on, but turn the other functions off. Apply number sense and systematic trial to set an appropriate viewing window.



CONNECTIONS

The probability that an event will occur is the total number of favourable outcomes divided by the total number of possible outcomes. You will learn more about probability if you study data management.

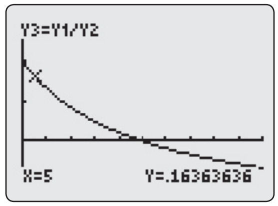


The combined function $f(g) = \frac{C(g)}{N(g)}$ represents the number of free caps randomly given out divided by the number of fans. Therefore, this function represents the probability that a fan will receive a free baseball cap as a function of the game number.

To determine the probability of receiving a free cap at game 5, evaluate $f(g) = \frac{C(g)}{N(g)}$ for $g = 5$.

Method 1: ***Use a Graphing Calculator***

Use the **Value** operation to identify the function value when $x = 5$.



According to the graph, there is about a 16% chance of receiving a free baseball cap at game 5.

Method 2: ***Use Pencil and Paper***

Substitute $g = 5$ into the equation for $f(g) = \frac{C(g)}{N(g)}$ and evaluate.

$$f(g) = \frac{2 - 0.04g}{10 + 0.2g}$$

$$\begin{aligned} f(5) &= \frac{2 - 0.04(5)}{10 + 0.2(5)} \\ &= \frac{2 - 0.2}{10 + 1} \\ &= \frac{1.8}{11} \\ &= 0.1\overline{63} \end{aligned}$$

According to the equation, there is about a 16% chance of receiving a free baseball cap at game 5.

KEY CONCEPTS

- A combined function of the form $y = f(x)g(x)$ represents the product of two functions, $f(x)$ and $g(x)$.
- A combined function of the form $y = \frac{f(x)}{g(x)}$ represents the quotient of two functions, $f(x)$ and $g(x)$, for $g(x) \neq 0$.
- The domain of the product or quotient of functions is the domain common to the component functions. The domain of a quotient function $y = \frac{f(x)}{g(x)}$ is further restricted by excluding any values that make the denominator, $g(x)$, equal to zero.
- Products and quotients of functions can be used to model a variety of situations.

Communicate Your Understanding

C1 A damped harmonic oscillator is an object whose motion is cyclic, with a decaying (decreasing) amplitude, such as a released pendulum or a child on a swing after a single push. The motion of a damped harmonic oscillator can be modelled by a function of the form $y(t) = A\sin(kt) \times 0.5^{ct}$, where y represents distance as a function of time, t, and A, k, and c are constants.

a) What two component functions are combined to form this function? Sketch the shape of each component function.

b) Sketch the shape of the combined function.

C2 Refer to Example 1.

a) Will the graph of $y = \dfrac{g(x)}{f(x)}$ also produce a rational function or will it have a different shape? Explain your answer.

b) Will there be any asymptotes or holes in the graph? Justify your answer.

C3 Refer to Example 2. Suppose the owners took a more aggressive approach to reducing the ticket price. Predict the effect this would have on each function.

a) $P(g)$ **b)** $N(g)$ **c)** $C(g)$

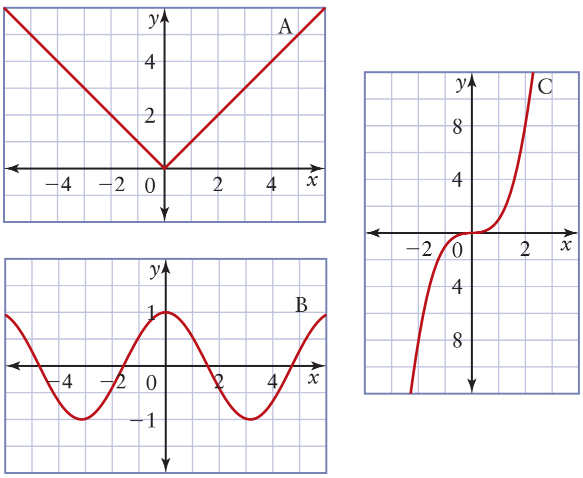
Justify your answers with mathematical reasoning.

A Practise

When graphing trigonometric functions, work in radians.

For help with questions 1 to 3, refer to the Investigate.

Use the graphs of the functions shown to answer questions 1 to 3.



1. a) Which two of these functions will multiply to form a combined function that is even? odd?

b) Is there more than one answer in either case in part a)? Explain.

2. Is the product of all three functions even, odd, or neither? Explain.

3. Use Technology Check your answers to questions 1 and 2 using graphing technology and the equations $y = |x|$, $y = \cos x$, and $y = x^3$.

For help with questions 4 and 5, refer to Example 1.

4. Let $f(x) = x - 2$ and $g(x) = x^2 - 4$. Develop an algebraic and a graphical model for each combined function. Then, give the domain and range of the combined function. Identify any holes or asymptotes.

a) $y = f(x)g(x)$ **b)** $y = \dfrac{f(x)}{g(x)}$ **c)** $y = \dfrac{g(x)}{f(x)}$

5. Let $u(x) = \cos x$ and $v(x) = 0.95^x$. Develop an algebraic and a graphical model for each combined function. Then, give the domain and range of the combined function. Identify any holes or asymptotes.

a) $y = u(x)v(x)$

b) $y = \dfrac{u(x)}{v(x)}$

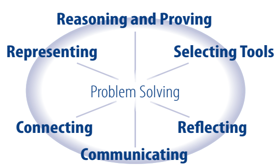
c) $y = \dfrac{v(x)}{u(x)}$

B Connect and Apply

Use the following information to answer questions 6 and 7. Graphing technology is recommended.

A fish pond initially has a population of 300 fish. When there is enough fish food, the population, P, of fish grows as a function of time, t, in years, as $P(t) = 300(1.05)^t$. The initial amount of fish food in the pond is 1000 units, where 1 unit can sustain one fish for a year. The amount, F, of fish food is decreasing according to the function $F(t) = 1000(0.92)^t$.

6. a) Graph the functions $P(t)$ and $F(t)$ on the same set of axes. Describe the nature of these functions.



b) Determine the mathematical domain and range of these functions.

c) Identify the point of intersection of these two curves. Determine the coordinates, to two decimal places, and explain what they mean. Call this point in time the crisis point.

d) Graph the function $y = F(t) - P(t)$. Explain the significance of this function.

e) What is the t-intercept, to two decimal places, of the function $y = F(t) - P(t)$? How does this relate to the crisis point?

f) Comment on the validity of the mathematical model for $P(t)$ for t-values greater than this intercept. Sketch how you think the curve should change in this region. Justify your answers.

For help with question 7, refer to Example 2.

7. a) Graph the function $y = \dfrac{F(t)}{P(t)}$ on a different set of axes. What does this function represent? Describe the shape of this function.

b) What are the coordinates of this function at the crisis point? Explain the meaning of these coordinates.

c) Describe the living conditions of the fish population before, at, and after the crisis point.

8. Let $f(x) = \sqrt{25 - x^2}$ and $g(x) = \sin x$.

a) Graph $f(x)$ and describe its shape. Is this function even, odd, or neither?

b) Graph $g(x)$ on the same set of axes. Is this function even, odd, or neither?

c) Predict the shape of $y = f(x)g(x)$. Sketch a graph of your prediction. Then, check your prediction using graphing technology.

d) Give the domain of $y = f(x)g(x)$. Estimate the range, to two decimal places, of $y = f(x)g(x)$. Explain why only an estimate is possible.

9. Refer to question 8.

a) Graph $y = \dfrac{g(x)}{f(x)}$. Is this function even, odd, or neither? Give the domain and range.

b) Graph $y = \dfrac{f(x)}{g(x)}$. Is this function even, odd, or neither? Give the domain and range.

Use the following information to answer questions 10 to 12.

Terra is a fictitious country. The population of Terra was 6 million in the year 2000 and is growing at a rate of 2% per year. The population, P, in millions, as a function of time, t, in years, can be modelled by the function $P(t) = 6(1.02)^t$.

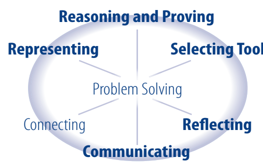
Terra's government has put in place economic plans such that food production is projected to follow the equation $F(t) = 8 + 0.04t$, where F is the amount of food that can sustain the population for 1 year.

10. a) Graph $P(t)$ and $F(t)$ on the same set of axes and describe their trends.

b) Graph the function $y = F(t) - P(t)$, and describe the trend. Does the Terrian nation currently enjoy a food surplus or suffer from a food shortage? Explain. What about in the years to come?

c) Identify the coordinates of the maximum of $y = F(t) - P(t)$ and explain what they mean.

11. The food production per capita is the amount of food production per person.



a) Graph the function $y = \dfrac{F(t)}{P(t)}$ and describe its trend. Use a separate grid, or adjust the viewing window to obtain a clear representation of the graph.

b) At what time is the food production per capita a maximum for Terra? Does this point coincide with the maximum that you found in question 10c)? Explain this result.

c) What does it mean when $\dfrac{F(t)}{P(t)} > 1$? $\dfrac{F(t)}{P(t)} < 1$? When are these conditions projected to occur for Terra?

12. Suppose that you are a cabinet minister in the Terrian government, with a portfolio in international trade and economic policy. What advice would you give your leader regarding projected food surpluses in the short to mid-term? What about shortages in the long term?

Use the following information to answer questions 13 and 14.

After the school dance on Thursday night, Carlos is thinking about asking Keiko out on a date, but his confidence is wavering. The probability, p_{Ask}, that Carlos will ask Keiko out as a function of time, t, in days following the dance, can be modelled by the function $p_{Ask}(t) = 0.45\sin t + 0.5$. Keiko, meanwhile, is interested in dating Carlos, but less so if he appears too eager or too shy. The probability, p_{Yes}, that Keiko will agree to a date if Carlos asks her, as a function of time, can be modelled by the function $p_{Yes}(t) = -0.08t(t - 7)$.

13. a) Graph the functions $p_{\text{Ask}}(t)$ and $p_{\text{Yes}}(t)$ on the same set of axes. Describe each trend. Determine the domain and range of each function in the context of this problem.

b) When is Carlos most likely to ask Keiko, to the nearest tenth of a day? Explain your answer.

c) When is Keiko most likely to agree to a date with Carlo if he asks her? Explain.

d) Graph the function $y = p_{\text{Ask}}(t) - p_{\text{Yes}}(t)$. Suggest what this graph may reveal.

14. The likelihood of Carlos and Keiko agreeing to a date is the conditional probability that Carlos asks Keiko and that, given that he asks, Keiko says yes. This conditional probability is given by the combined function $y = p_{\text{Ask}}(t)p_{\text{Yes}}(t)$.

a) Develop an algebraic and a graphical model for this combined function.

b) After what amount of time, to the nearest tenth of a day, is it most likely that Carlos and Keiko will agree to a date? What is the probability that this will happen at this time, to two decimal places?

c) At what point does Carlos have no chance of dating Keiko?

d) Suppose that you are friends with both Carlos and Keiko. What advice would you give to Carlos?

15. Does the product of functions commute? A mathematical process is commutative if you can reverse the order of the operands and obtain the same result. Pick any two functions, $f(x)$ and $g(x)$, and see if $f(x)g(x) = g(x)f(x)$. Try several examples and summarize your findings.

CONNECTIONS

The commutative property holds over the real numbers for the multiplication of two numbers.
$2 \times (-4) = (-4) \times 2$
$a \times b = b \times a, a, b \in \mathbb{R}$

16. Refer to question 15. Repeat the analysis for the division of two functions.

17. Given the functions $f(x) = 2^{-x}$ and $g(x) = x^3$, sketch a graph of the function $y = f(x)g(x)$ and describe its key features. Explain its end behaviour.

Achievement Check

18. Let $f(x) = x + 2$ and $g(x) = x^2 + 5x + 6$.

a) Determine each combined function.

i) $y = f(x)g(x)$

ii) $y = \dfrac{f(x)}{g(x)}$

b) State the domain and range of $y = \dfrac{f(x)}{g(x)}$.

C Extend and Challenge

19. a) Given an even function $f(x)$, what is the symmetrical behaviour of a combined function that is

i) the square of $f(x)$, $[f(x)]^2$?

ii) the cube of $f(x)$, $[f(x)]^3$?

iii) the nth power of $f(x)$, $[f(x)]^n$?

Use algebraic reasoning with examples to justify your answers.

b) Repeat the analysis for powers of odd functions.

c) **Use Technology** Use graphing technology to verify your results.

20. The motion of a pendulum can be modelled by the function $x(t) = 10\cos(2t) \times 0.95^t$, where x is the horizontal displacement from the rest position, in centimetres, as a function of time, t, in seconds.

a) Determine the length of the pendulum.

b) Sketch how the shape of the graph would change if

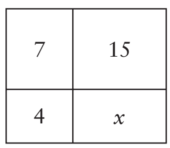
i) the air resistance were reduced

ii) the pendulum were lengthened

Justify your answers with mathematical reasoning.

21. **Math Contest** Increasing x by $y\%$ gives 12. Decreasing x by $y\%$ gives 8. Determine the value of x.

22. **Math Contest** A rectangle is divided into four smaller rectangles with areas of 4, 7, 15, and x, as shown. Determine the value of x.



23. **Math Contest** Given that $f(xy) = \dfrac{f(x)}{y}$ for all real numbers x and y and $f(500) = 3$, find $f(600)$.

24. **Math Contest** A rectangular container with base 9 cm by 11 cm has height 38.5 cm. Assuming that water expands 10% when it freezes, determine the depth to which the container can be filled so that the ice does not rise above the top of the container when the water freezes.

8.3 Composite Functions

You have learned four ways of combining functions—sums, differences, products, and quotients. Another type of combined function occurs when one function depends on another. For example, in a predator-prey relationship, the predator population is likely to depend on the prey population as its primary food source, which may fluctuate over time. Think about the effect on a predator population when its food source is plentiful versus when it is scarce.

Investigate — How can the composition of functions be used to model a predator-prey relationship?

Tools

- graphing calculator or graphing software

1. The number, R, of rabbits in a wildlife reserve as a function of time, t, in years can be modelled by the function $R(t) = 50\cos t + 100$.

a) Graph this function. Describe the trend.

b) Identify the following key features and their meanings.

i) domain **ii)** range **iii)** period **iv)** frequency

2. The number, W, of wolves in the same reserve as a function of time can be modelled by the function $W(t) = 0.2[R(t - 2)]$.

a) Graph the function $W(t)$ on the same set of axes as in step 1. Adjust the viewing window to focus on this function.

b) Identify the following key features and explain their meanings.

i) domain **ii)** range **iii)** period **iv)** frequency

3. Using the equation from step 1, substitute $R(t - 2)$ into the equation for $W(t)$ and develop a simplified equation to model the number of wolves as a function of time.

4. **Reflect**

a) Set the viewing window so that you can see the graphs of both $R(t)$ and $W(t)$ simultaneously. Compare the graphs, and explain how they are

i) alike

ii) different

b) Suggest possible reasons for the relationship between these two functions.

Technology Tip

If you are using *The Geometer's Sketchpad®*, you can graph one function in terms of another. For example, if the rabbit function is $f(x)$, you can graph the wolf population by entering $0.2*f(x - 2)$.

If you are using a graphing calculator, press VARS and use the **Y-VARS** secondary menu to enter Y2 = 0.2*(Y1(X − 2)).

A **composite function** is a function that depends on another function. It is formed when one function is substituted into another.

> $f(g(x))$ is the composite function that is formed when $g(x)$ is substituted for x in $f(x)$.
> $f(g(x))$ is read as "f of g of x."
>
> Note that the order of the functions is important. As read from left to right, the second function is substituted into the first function. $(f \circ g)(x)$ is an alternative notation that means the same as $f(g(x))$.

Example 1 Determine the Composition of Two Functions

Let $f(x) = x^2$ and $g(x) = x + 3$. Determine an equation for each composite function, graph the function, and give its domain and range.

a) $y = f(g(x))$ **b)** $y = g(f(x))$ **c)** $y = f(f(x))$
d) $y = g(g(x))$ **e)** $y = g^{-1}(g(x))$

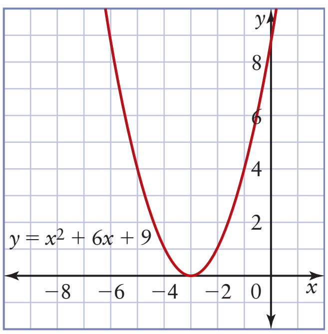
Solution

a) To determine the composite function $y = f(g(x))$, substitute the function $g(x)$ for x in the function $f(x)$.

$$\begin{aligned} f(x) &= x^2 \\ f(g(x)) &= (g(x))^2 \\ &= (x + 3)^2 \\ &= x^2 + 6x + 9 \end{aligned}$$

Expand the perfect square trinomial.

This is a quadratic function. There are no restrictions on the value of x in this equation. Therefore, the domain is $x \in \mathbb{R}$. This can be verified by inspecting the graph. Looking at the factored form, $f(g(x)) = (x + 3)^2$, the graph of this function is a horizontal translation of 3 units to the left of the graph of $y = x^2$.

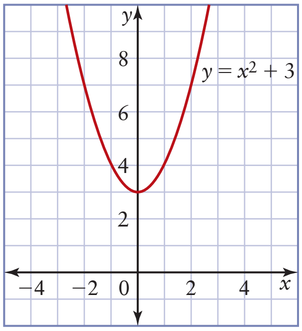


From the graph, it is clear that the range is $\{y \in \mathbb{R}, y \geq 0\}$.

b) To determine the composite function $y = g(f(x))$, substitute the function $f(x)$ for x in the function $g(x)$.

$$\begin{aligned} g(x) &= x + 3 \\ g(f(x)) &= (f(x)) + 3 \\ &= (x^2) + 3 \\ &= x^2 + 3 \end{aligned}$$

This is a quadratic function. Its graph can be generated by vertically translating the graph of $y = x^2$ up 3 units.



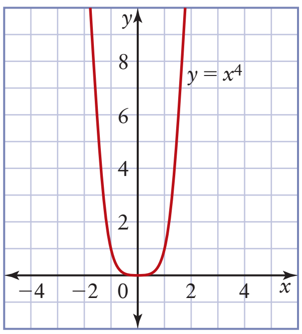
The domain of this function is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}, y \geq 3\}$.

Note that the different results for parts a) and b) suggest that the order of the component functions of a composite function is important: $f(g(x)) \neq g(f(x))$.

c) To determine the composite function $y = f(f(x))$, substitute the function $f(x)$ for x in the function $f(x)$.

$$\begin{aligned} f(x) &= x^2 \\ f(f(x)) &= (f(x))^2 \\ &= (x^2)^2 \\ &= x^4 \end{aligned}$$

This is a quartic power function, and its graph is shown.



The domain of this function is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.

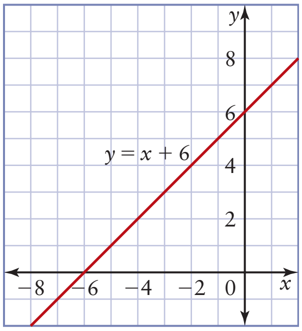
CONNECTIONS

You studied power functions in Chapter 1.

d) To determine the composite function $y = g(g(x))$, substitute the function $g(x)$ for x in the function $g(x)$.

$$\begin{aligned} g(x) &= x + 3 \\ g(g(x)) &= (g(x)) + 3 \\ &= (x + 3) + 3 \\ &= x + 6 \end{aligned}$$

This is a linear function.



The domain of this function is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.

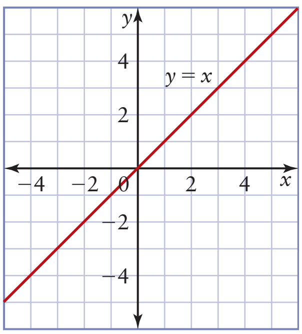
e) To determine an expression for $y = g^{-1}(g(x))$, it is necessary to determine the inverse function $y = g^{-1}(x)$.

$y = x + 3$	Write $g(x)$ in terms of x and y.
$x = y + 3$	Interchange x and y.
$x - 3 = y$	Isolate y.
$y = x - 3$	

Therefore, the inverse function is $g^{-1}(x) = x - 3$. To determine $g^{-1}(g(x))$, substitute $g(x)$ into $g^{-1}(x)$.

$$\begin{aligned} g^{-1}(g(x)) &= (x + 3) - 3 \\ &= x + 3 - 3 \\ &= x \end{aligned}$$

This is a linear function.



The domain of this function is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.

Example 2 Evaluate a Composite Function

Let $u(x) = x^2 + 3x + 2$ and $w(x) = \frac{1}{x - 1}$.

a) Evaluate $u(w(2))$.

b) Evaluate $w(u(-3))$.

Solution

a) Method 1: ***Determine u(w(x)) and Then Substitute***

Determine an expression for $u(w(x))$.

$$u(w(x)) = \left(\frac{1}{x - 1}\right)^2 + 3\left(\frac{1}{x - 1}\right) + 2$$

Substitute $x = 2$.

$$\begin{aligned} u(w(2)) &= \left(\frac{1}{2 - 1}\right)^2 + 3\left(\frac{1}{2 - 1}\right) + 2 \\ &= 1^2 + 3(1) + 2 \\ &= 6 \end{aligned}$$

Method 2: ***Determine w(2) and Then Substitute***

Determine $w(2)$.

$$\begin{aligned} w(x) &= \frac{1}{x - 1} \\ w(2) &= \frac{1}{2 - 1} \\ &= 1 \end{aligned}$$

Substitute $w(2) = 1$ into $u(x)$.

$$\begin{aligned} u(w(2)) &= u(1) \\ &= 1^2 + 3(1) + 2 \\ &= 6 \end{aligned}$$

b) Determine $u(-3)$. Then, substitute the result into $w(x)$ and evaluate.

$$\begin{aligned} u(x) &= x^2 + 3x + 2 \\ u(-3) &= (-3)^2 + 3(-3) + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} w(u(-3)) &= w(2) \\ &= \frac{1}{2 - 1} \\ &= 1 \end{aligned}$$

Example 3 Music Sales Revenue

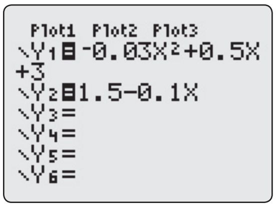
A music store has traditionally made a profit from sales of CDs and cassette tapes. The number, C, in thousands, of CDs sold yearly as a function of time, t, in years since the store opened, can be modelled by the function $C(t) = -0.03t^2 + 0.5t + 3$. The number, T, in thousands, of cassette tapes sold as a function of time can be modelled by the function $T(t) = 1.5 - 0.1t$. The store opened in 1990, at which point $t = 0$.

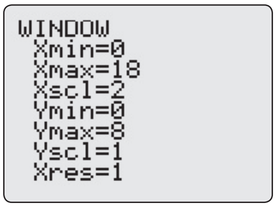
a) Graph both functions up to the year 2008 and describe their trends.

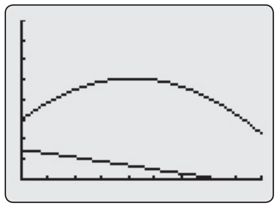
b) The total revenue, R, from sales of CDs and cassette tapes can be modelled by the composite function $R(t) = [3C(t) + 2T(t)](1.04^t)$. Develop an algebraic and a graphical model for the store's revenue, and interpret the trend.

Solution

a) Use a graphing calculator. Apply number sense and systematic trial to set an appropriate viewing window.

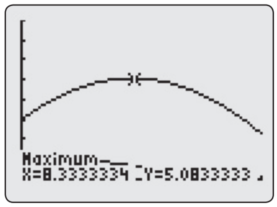






The graph indicates that the number of CDs sold increased from around 3000 per year to just over 5000 per year around 1999, after which it declined. To identify when this happened, use the **Maximum** operation.

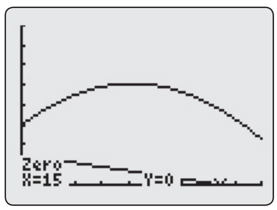
Press [2nd] [CALC], select **4:maximum**, and follow the prompts.



About a third of the way through 1998, CD sales per year reached about 5083. Then, sales began decreasing.

The number of cassette tapes sold has declined steadily from 1500 per year. In fact, the graph shows that the store stopped selling tapes after a certain time. To identify when this happened, use the **Zero** operation.

Press [2nd] [CALC], select **2:zero**, and follow the prompts.



Tape sales ceased when $t = 15$, which corresponds to the year 2005 $(1990 + 15 = 2005)$.

b) To develop the revenue function, substitute the CD and cassette tape sales functions into the revenue equation and simplify.

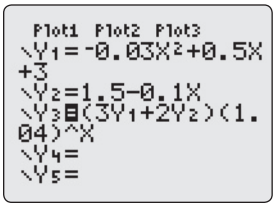
$$R(t) = [3C(t) + 2T(t)](1.04^t)$$
$$= [3(-0.03t^2 + 0.5t + 3) + 2(1.5 - 0.1t)](1.04^t)$$
$$= (-0.09t^2 + 1.5t + 9 + 3 - 0.2t)(1.04^t) \quad \text{Apply the distributive property.}$$
$$= (-0.09t^2 + 1.3t + 12)(1.04^t) \quad \text{Collect like terms.}$$

This simplified revenue function is the product of a quadratic function and an exponential function. To illustrate this, graph the revenue function on a new set of axes.

Method 1: ***Graph the Composite Function in Terms of C(n) and T(n)***

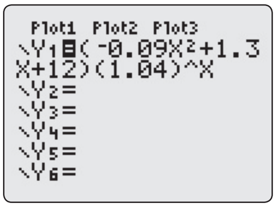
Press VARS and cursor over to the **Y-VARS** menu to access **Y1** and **Y2**.

Turn off **Y1** and **Y2** to view **Y3** alone.

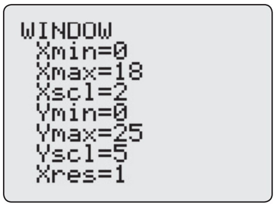


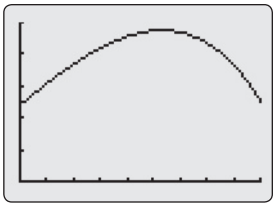
Method 2: ***Graph the Simplified Revenue Function***

Clear all functions and enter the simplified revenue function.

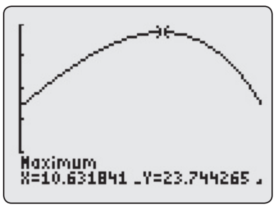


Use the window settings shown.





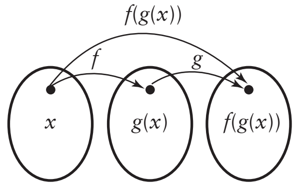
The trend in revenue from music sales steadily increased throughout the 1990s, peaking around the year 2001, after which revenues began to sharply decline.



Maximum revenues from sales of almost $24 000 were achieved in the eleventh year of business, for the year 2000–2001.

KEY CONCEPTS

- $f(g(x))$ denotes a composite function, that is, one in which the function $f(x)$ depends on the function $g(x)$. This can also be written as $(f \circ g)(x)$.

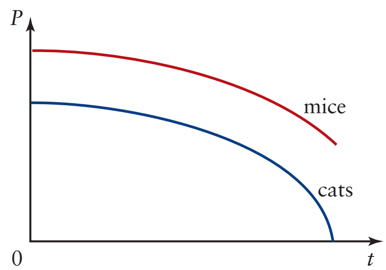


- To determine an equation for a composite function, substitute the second function into the first, as read from left to right. To determine $f(g(x))$, substitute $g(x)$ for x in $f(x)$.
- To evaluate a composite function $f(g(x))$ at a specific value, substitute the value into the equation of the composite function and simplify, or evaluate $g(x)$ at the specific value and then substitute the result into $f(x)$.

Communicate Your Understanding

C1 Does $f(g(x))$ mean the same thing as $f(x)g(x)$? Explain, using examples to illustrate.

C2 A predator-prey relationship is shown. Population levels are shown as functions of time.



a) Is this a stable relationship? Explain why or why not.

b) What has happened to the cat population? Suggest some reasons why this may have happened.

c) Predict what will happen to the mouse population in the future. Justify your answer.

C3 Refer to the revenue function in Example 3, $R(t) = [3C(t) + 2T(t)](1.04^t)$.

a) What do the terms $3C(t)$ and $2T(t)$ represent?

b) Suggest a reason why the factor of 1.04^t is present.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. Let $f(x) = -x + 2$ and $g(x) = (x + 3)^2$. Determine a simplified algebraic model for each composite function.

a) $y = f(g(x))$ **b)** $y = g(f(x))$

c) $y = f(f(x))$ **d)** $y = g(g(x))$

e) $y = f^{-1}(f(x))$

2. Graph each composite function in question 1. Give the domain and range.

3. Use Technology Check your answers to question 2 using graphing technology.

For help with question 4, refer to Example 2.

4. Let $f(x) = x^2 + 2x - 4$ and $g(x) = \dfrac{1}{x + 1}$.

a) Evaluate $g(f(0))$.

b) Evaluate $f(g(-2))$.

c) Show that $g(f(x))$ is undefined for $x = 1$ and $x = -3$.

B Connect and Apply

For help with questions 5 to 7, refer to Example 3.

Use the following information to answer questions 5 to 7.

In an election campaign, the popularity, B, as a percent of voters, of the governing Blue party can be modelled by a function of time, t, in days throughout the campaign as B(t) = 40 − 0.5t. The popularity, R(t), of the opposing Red party can be modelled by a composite function of B(t), R(B(t)) = 20 + 0.75[40 − B(t)].

5. a) Graph $B(t)$ and describe the trend.

b) What is the popularity of the Blue party at the beginning of the campaign?

c) What is the rate of change of this function, and what does it mean?

d) Can you tell if these are the only two parties in the election? If you can, explain how. If you cannot, describe what additional information is required.

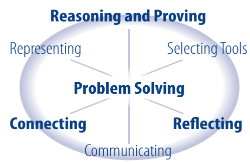
6. a) Graph $R(t)$ and describe the trend.

b) What is the popularity of the Red party at the beginning of the campaign?

c) What is the rate of change of this graph, and what does it mean?

d) If it can be assumed that these are the only two parties running for election, which party do you think will win? Does your answer depend on something? Explain.

7. Assume that there are at least three parties running in the election.



a) Graph the composite function $V(t) = 100 - [(B(t) + R(t)]$. What does this graph represent? Describe the trend.

b) Assuming that this is a three-party election, and that there are no undecided votes, can you tell who will win this election? Explain.

c) Repeat part b) assuming that there are four parties running.

8. Is $f(g(x)) = g(f(x))$ true for all functions $f(x)$ and $g(x)$? Justify your answer, using examples to illustrate.

9. Let $f(x) = x^3$.

a) Determine $f^{-1}(x)$.

b) Determine $f(f^{-1}(x))$.

c) Determine $f^{-1}(f(x))$.

d) Compare your answers to parts b) and c). Describe what you notice.

e) Determine $f(f^{-1}(3))$, $f(f^{-1}(5))$, and $f(f^{-1}(-1))$. What do you notice?

10. Refer to question 9. Does this result apply to all functions? Design and carry out an investigation to determine the answer. Use functions whose inverses are also functions. Write a conclusion based on your findings.

11. Let $f(x) = x^2$, $g(x) = x^3$, and $h(x) = \sin x$. Work in radians.

a) Predict what the graph of $y = f(h(x))$ will look like, and sketch your prediction.

b) **Use Technology** Check your prediction using graphing technology.

c) Is the function in part a) periodic? Explain.

d) Identify the domain and range.

12. a) Repeat question 11 for $y = g(h(x))$.

b) Compare $y = f(h(x))$ and $y = g(h(x))$. How are these functions similar? different?

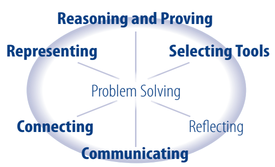
13. An environmental scientist measures the presence of a pollutant in a lake and models the concentration, C, in parts per million (ppm), as a function of the population, P, of the lakeside city, using the function $C(P) = 1.54P + 58.55$. The city's population, in thousands, can be modelled by the function $P(t) = 12.1 \times 2^{\frac{t}{52}}$, where t is the time, in years.

a) Determine an equation for the concentration of pollutant as a function of time.

b) Sketch a graph of this function.

c) How long will it take for the concentration to reach 100 ppm, to the nearest year?

14. Is $(f \circ g)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$, or is $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$, or is neither true? Verify using two examples involving at least two different types of functions.



15. **Chapter Problem** Funky Stuff is releasing a new stuffed bear called "Funky Teddy Bear" in time for the winter holiday shopping rush. The number, N, in thousands, of Funky Teddy Bears sold is projected to be a function of the price, p, in dollars, as $N(p) = -0.1(p + 5)^2 + 80$. Revenue, R, from sales is given by the product of number sold and price: $R(p) = N(p) \times p$. Your task is to determine the optimum price that will maximize revenues.

a) Graph $N(p)$. Describe and interpret the trend.

b) Assuming a minimum price of \$5, what price domain will yield revenue?

c) Can you tell the optimum price from this graph? Explain why or why not.

d) Graph the function $R(p)$.

e) At what price should Funky Stuff sell Funky Teddy Bears? How much revenue will this bring?

16. A manufacturer models its weekly production of office chairs since 2001 by the function $N(t) = 100 + 25t$, where t is the time, in years, since 2001, and N is the number of chairs. The size of the manufacturer's workforce can be modelled by the composite function $W(N) = 3\sqrt{N}$.

a) Write the size of the workforce as a function of time.

b) State the domain and range of the new function and sketch its graph.

Achievement Check

17. Let $f(x) = x^2$, $g(x) = x - 2$, and $h(x) = \frac{1}{x}$.

a) Determine a simplified algebraic model for each composite function.

i) $f(g(x))$

ii) $h(g(x))$

iii) $g^{-1}(h(x))$

b) Evaluate $f(h(2))$.

C Extend and Challenge

18. Let $f(x) = \log x$, $g(x) = \sin x$, and $h(x) = \cos x$. Work in radians.

a) What is the domain of $f(x)$?

b) Use this information to predict the shape of the graph of the composite function $y = f(g(x))$. Sketch your prediction.

c) **Use Technology** Check your prediction in part b) using graphing technology. Give the domain and range of $y = f(g(x))$.

d) Use your result in part c) to predict the shape of the graph of the composite function $y = f(h(x))$. Sketch your prediction.

e) **Use Technology** Check your prediction in part d) using graphing technology. Give the domain and range of $y = f(h(x))$.

19. Let $f(x) = x^2 - 9$ and $g(x) = \frac{1}{x}$.

a) Determine the domain and range of $y = f(g(x))$.

b) Determine the domain and range of $y = g(f(x))$.

c) **Use Technology** Use graphing technology to verify your answers in parts a) and b).

20. Two ships leave port at the same time. The first ship travels north at 4 km/h and the second ship travels east at 5 km/h. Write their distance apart as a function of time.

21. **Math Contest** Given that $f(x) = \frac{ax + b}{cx + d}$, find $f^{-1}(x)$.

22. **Math Contest** The mean of the nine numbers in the set {9, 99, 999, . . . , 999 999 999} is a nine-digit number n, all of whose digits are distinct. The number n does *not* contain what digit?

23. **Math Contest** Find all real numbers x such that $\sqrt{1 - \sqrt{1 - x}} = x$.

24. **Math Contest** The sum of two numbers is 7 and their product is 25. Determine the sum of their reciprocals.

25. **Math Contest** A number is called a decreasing number if each digit is less than the preceding digit. How many three-digit decreasing numbers are less than 500?

CAREER CONNECTION

After completing a 4-year bachelor of mathematics degree in actuarial sciences at the University of Waterloo, Farah works as an actuarial student for a large insurance agency. At the same time, she is studying for exams so that she can become a fully qualified actuary. In her job, Farah deals with probabilities and statistics, so that she can determine how much the insurance company should charge for its automobile insurance policies. Before the company insures an automobile, Farah helps to determine the risk of collisions. Her goal is to make sure that the policy is sold for a fair price that will be enough to pay claims and allow the company to make a profit.

8.4

Inequalities of Combined Functions

Suppose that you are a financial advisor for a building contractor. Your client has been offered a contract to build up to 500 new houses but is not sure how many she should build. If she builds too few, she might not make enough money to cover expenses. If she builds too many, market saturation might drive down prices, resulting in the risk of a loss. Is there an optimum number that you would advise your client to build?

Often, in business, as well as in other disciplines, analysing combined functions can lead to a range of acceptable solutions. In such cases, techniques for solving inequalities are often useful. Deeper analysis can also reveal conditions for an optimum solution.

Investigate — If you build it, will they come?

Tools

- graphing calculator and/or grid paper

A building developer can build up to 500 houses for a new subdivision for the following costs:

Fixed Cost: \$8 000 000
Variable Cost: \$65 000 per house

The developer earns revenue, R, in millions of dollars, from sales according to the function $R(n) = 1.6\sqrt{n}$, where n is the number of houses.

1. Write an equation to represent the total cost, C, in millions of dollars, as a function of the number, n, of houses built and sold.

2. Graph $R(n)$ and $C(n)$ on the same set of axes. Choose appropriate window settings so that sales of between 0 and 500 houses can be viewed.

3. Identify the region where $R(n) > C(n)$. Sketch the graph and shade in this region. What is its significance?

4. Identify the points of intersection of the two curves and determine their coordinates using a method of your choice. Explain the meaning of these coordinates.

5. **a)** Use the superposition principle to graph $y = R(n) - C(n)$. Adjust the viewing window, as needed.

b) Describe at least three interesting things that this graph shows.

6. **Reflect** Write a brief letter to the housing developer, advising her about the subdivision contract. Include in your letter
- the minimum and maximum number of houses that she should agree to build
- the number of houses she would need to build and sell to yield a maximum profit
- suggestions on how she might be able to increase her profits

Support your advice with mathematical reasoning.

Example 1 Techniques for Illustrating Inequalities

Let $f(x) = x$ and $g(x) = (x - 2)^2$.

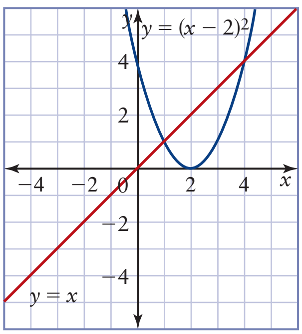
a) Graph the functions on the same set of axes. Identify the points of intersection.

b) Illustrate the regions for which

i) $f(x) > g(x)$ **ii)** $g(x) > f(x)$

Solution

a) The graph of $y = f(x)$ is a line with slope one, passing through the origin. The graph of $y = g(x)$ can be obtained by applying a horizontal translation of 2 units to the right of the graph of $y = x^2$.



From the graph, the points of intersection appear to be (1, 1) and (4, 4). This can be verified algebraically by solving the linear-quadratic system of $f(x)$ and $g(x)$. Set the two functions equal and solve for x.

$$\begin{aligned} f(x) &= g(x) \\ x &= (x - 2)^2 \\ x &= x^2 - 4x + 4 \\ x^2 - 5x + 4 &= 0 \\ (x - 1)(x - 4) &= 0 \end{aligned}$$

$$x - 1 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 1 \quad \text{or} \quad x = 4$$

To verify that the points of intersection are (1, 1) and (4, 4), mentally substitute each of these ordered pairs into $f(x) = x$ and $g(x) = (x - 2)^2$.

> **CONNECTIONS**
>
> You used interval notation in Chapter 1. Recall that (1, 4) means $1 < x < 4$.

b) **Method 1:** ***Compare the Functions Visually***

Using different colours or line styles is a useful technique for visualizing regions for which one function is greater than another. The graph of $y = f(x)$ is above the graph of $y = g(x)$ between $x = 1$ and $x = 4$. Therefore,

i) $f(x) > g(x)$ on the interval (1, 4)

ii) $g(x) > f(x)$ on the intervals $(-\infty, 1) \cup (4, \infty)$

Method 2: ***Analyse the Difference Function***

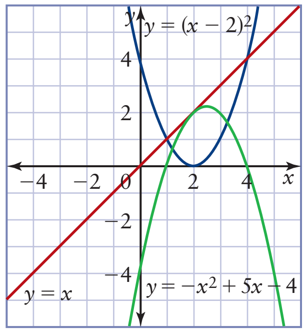
Another way to illustrate where one function is greater than another is to subtract the functions and see where the graph of the difference is above the x-axis.

$f(x) > g(x)$

$f(x) - g(x) > 0$ Subtract $g(x)$ from both sides.

$$\begin{aligned} f(x) - g(x) &= x - (x - 2)^2 \\ &= x - (x^2 - 4x + 4) \\ &= -x^2 + 5x - 4 \end{aligned}$$

This function, $y = -x^2 + 5x - 4$, is positive on the intervals where $f(x) - g(x) > 0$ and therefore where $f(x) > g(x)$.



This function is positive on the interval (1, 4). Therefore,

i) $f(x) > g(x)$ on the interval (1, 4)

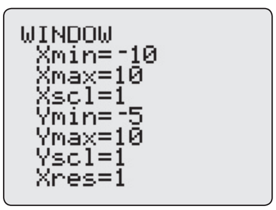
ii) $g(x) > f(x)$ on the intervals $(-\infty, 1) \cup (4, \infty)$

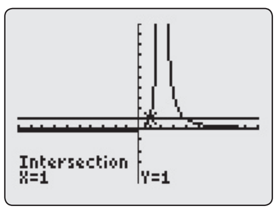
Method 3: ***Analyse the Quotient Function***

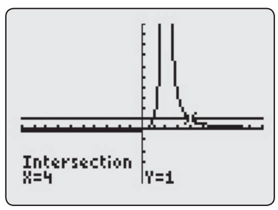
The functions $f(x)$ and $g(x)$ can also be compared by analysing their quotient. Graph the combined function $\frac{f(x)}{g(x)} = \frac{x}{(x - 2)^2}$ and identify the interval(s) for which this quotient is greater than one, which will correspond to where $f(x) > g(x)$.

To make this more clearly visible, graph the combined function using a graphing calculator with an appropriate viewing window. Include a graph of $y = 1$. Use the **Intersect** operation to identify the coordinates of the points of intersection.

> **CONNECTIONS**
>
> A fraction is greater than one when the numerator is greater than the denominator:
>
> $\frac{7}{6} > 1$
>
> Likewise, a fraction is less than one when the numerator is less than the denominator:
>
> $\frac{5}{6} > 1$
>
> The same reasoning can be applied to a quotient of functions.







It appears that $\frac{f(x)}{g(x)} > 1$ on the interval (1, 4). However, $\frac{f(x)}{g(x)}$ is not defined for $x = 2$. Determine $f(2)$ and $g(2)$ to decide what happens when $x = 2$.

$f(2) = 2$ and $g(2) = 0$.

Thus, $f(2) > g(2)$.

Therefore,

i) $f(x) > g(x)$ on the interval (1, 4)

ii) $g(x) > f(x)$ on the intervals $(-\infty, 1) \cup (4, \infty)$

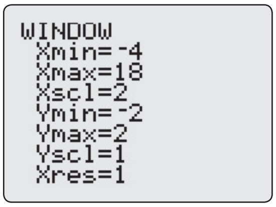
Example 2 Solve an Inequality Graphically

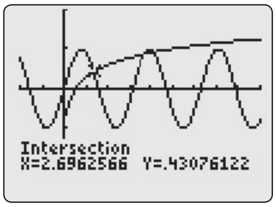
Solve $\sin x > \log x$.

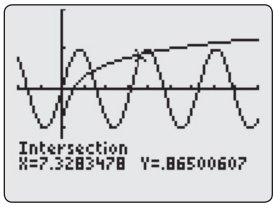
Solution

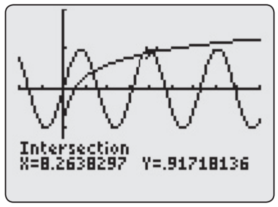
The solutions to this inequality cannot be found by algebraic techniques.

Graph $y = \sin x$ and $y = \log x$ on a graphing calculator. Ensure that the calculator is set to Radian mode. Use the window settings shown. Use the **Intersect** operation to find the approximate x-coordinates of any points of intersection.





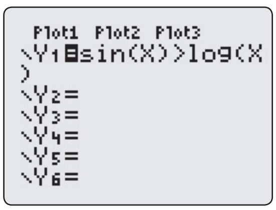




There is an asymptote at $x = 0$ and points of intersection at approximately (2.70, 0.43), (7.33, 0.87), and (8.26, 0.92). The graph of $y = \sin x$ is above the graph of $y = \log x$ between $x = 0$ and $x \doteq 2.70$ and between $x \doteq 7.33$ and $x \doteq 8.26$. Therefore, $\sin x > \log x$ on the approximate intervals $(0, 2.70) \cup (7.33, 8.26)$.

You can verify this using the **TEST** menu of a graphing calculator.

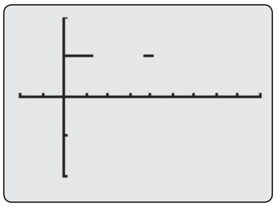
- Enter the inequality in the **Y=** editor. For >, press [2nd] [TEST] to access the **TEST** menu and select **3: >**.



Technology Tip

To determine if the endpoints of an interval are included, press TRACE and move the cursor to each endpoint. If the calculator displays a value for y, then the endpoint is included. If the y-value is blank, then the endpoint is not included.

- Press MODE and set the calculator to Dot mode.
- Press GRAPH.



The graphing calculator plots the y-value 1 when the inequality is true and the y-value 0 when it is false. From the screen, you can see that $\sin x > \log x$ on the approximate intervals $(0, 2.70) \cup (7.33, 8.26)$.

Example 3 Inventory Control

A computer store's cost, C, for shipping and storing n computers can be modelled by the function $C(n) = 1.5n + \frac{200\ 000}{n}$. The storage capacity of the store's warehouse is 750 units.

a) Graph this function and explain its shape. What is the domain of interest for this problem?

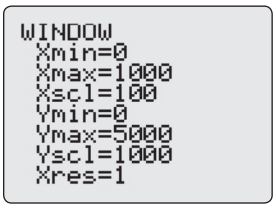
b) Determine the minimum and maximum number of computers that can be ordered at any one time to keep costs below \$1500, assuming that inventory has fallen to zero.

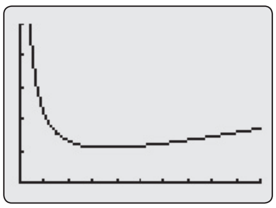
c) What is the optimum order size that will minimize storage costs?

d) Why might this not be the best number to order?

Solution

a) Use a graphing calculator. Apply number sense to choose appropriate window settings.





The shape of this graph can be understood if it is thought of as the superposition of a linear function and a rational function.

$$C(n) = 1.5n + \frac{200\ 000}{n}$$

linear (1.5n) rational (200 000/n)

The domain of $C(n)$ is $\{n \in \mathbb{R}, n > 0\}$.

b) Determine the values of n that will keep costs below \$1500.

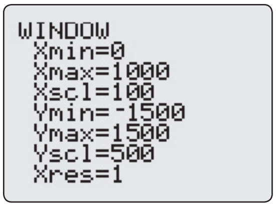
$$C(n) < 1500$$

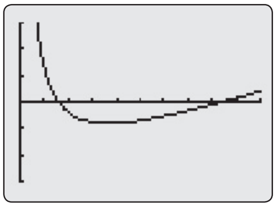
$$1.5n + \frac{200\,000}{n} < 1500$$

$$1.5n + \frac{200\,000}{n} - 1500 < 0$$

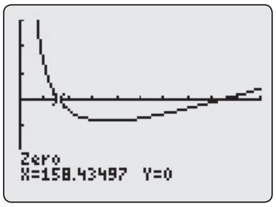
Method 1: *Use a Graphing Calculator*

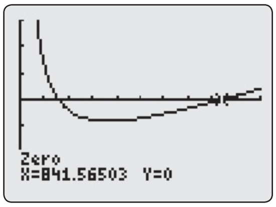
Graph the adjusted cost function $C = 1.5n + \frac{200\,000}{n} - 1500$ and identify the region below the horizontal axis. Apply number sense to choose appropriate window settings to view the region of interest.



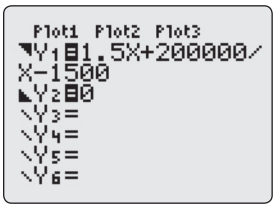


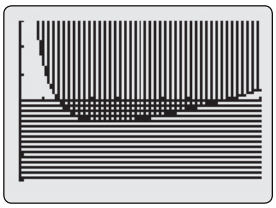
Notice that the adjustment term translates the original cost curve down 1500 units, making it easy to distinguish the region of interest. Locate the zeros by using the **Zero** operation.





According to the graph, an inventory order of from 159 to 841 units will keep costs below \$1500. This region can be illustrated by setting **Y2** = 0 and then choosing shading above and below for the line styles of **Y1** and **Y2**, respectively.





The region where the two shaded parts overlap shows the interval for which costs are below \$1500.

Method 2: *Use Pencil and Paper*

$1.5n^2 + 200\,000 - 1500n < 0$ Multiply both sides by n, since $n > 0$.

$1.5n^2 - 1500n + 200\,000 < 0$ Write the quadratic expression in standard form.

The quadratic expression on the left side corresponds to a quadratic function that opens upward. The region between the zeros will correspond to n-values for which $C < 0$. Solve for these zeros.

$1.5n^2 - 1500n + 200\,000 = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Apply the quadratic formula.

$$= \frac{-(-1500) \pm \sqrt{(-1500)^2 - 4(1.5)(200\,000)}}{2(1.5)}$$

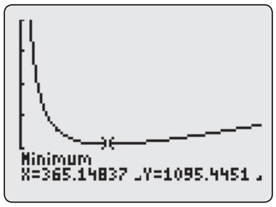
$$= \frac{1500 \pm \sqrt{1\,050\,000}}{3}$$

$$\doteq 158 \text{ or } 842$$

Since the warehouse has a maximum storage capacity of 750 units, a shipment of from 159 to 750 computers will keep costs below \$1500.

c) To determine the optimum order size that will minimize shipping and storage costs, locate the minimum of the original cost function, $C(n) = 1.5n + \frac{200\,000}{n}$.

Use the **Minimum** operation of a graphing calculator.



The minimum cost occurs when approximately 365 computers are ordered. The corresponding cost for this size of order is approximately \$1095.

d) The optimum order size may not be desirable, particularly if consumer demand is not aligned with this value. If demand significantly exceeds 365 units, then profit will be lost due to unfulfilled sales requests. On the other hand, if demand is significantly below 365 units, unnecessary shipping and storage costs will occur, as well as the cost of lost inventory space for other products.

KEY CONCEPTS

- Solutions to problems involving combined functions can sometimes lead to a range of acceptable answers. When this happens, techniques for solving inequalities are applied.
- There are a number of ways to graphically illustrate an inequality involving a combined function.
- Algebraic and graphical representations of inequalities can be useful for solving problems involving combined functions.

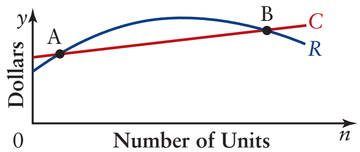
Communicate Your Understanding

C1 **a)** Refer to Example 1. List and summarize the methods for illustrating the inequality of two functions.

b) Describe at least one advantage and one disadvantage of each technique.

Use the following information to answer questions C2 and C3.

The cost, C, and revenue, R, functions for a business venture are shown on the graph. Assume that the independent variable is the number of units.



C2 **a)** Copy the graph. Shade the region where $R > C$. What is the meaning of this region?

b) Shade the region where $R < C$, using a different colour or shading style. What is the meaning of this region?

C3 **a)** Sketch a graph of the combined function $y = R - C$.

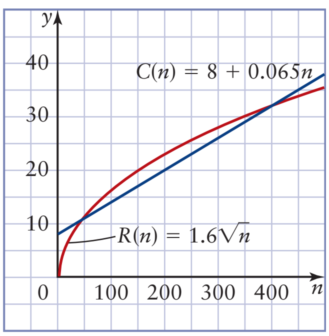
b) How would this graph change if the slope of the cost function were increased? decreased? What does each of these scenarios imply about potential profit?

A Practise

For help with questions 1 and 2, refer to the Investigate.

Use the following information to answer questions 1 and 2.

The revenue and cost functions for the housing developer are shown on the graph.



1. **a)** Suppose that the developer found a way to reduce her variable cost to \$58 000 per house. How would this affect

i) the minimum and maximum number of houses she could build?

ii) her maximum potential profit?

b) Sketch a graph to illustrate your explanation.

2. **a)** Suppose that, instead of her variable costs being reduced, her fixed costs increase by \$2 000 000. How would this affect

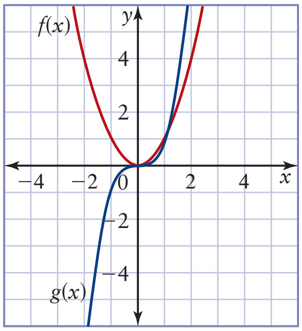
i) the minimum and maximum number of houses she could build?

ii) her maximum potential profit?

b) Sketch a graph to illustrate your explanation.

For help with questions 3 to 7, refer to Example 1.

3. The graphs of two functions are shown.



a) For what values of x is

i) $f(x) > g(x)$?

ii) $f(x) < g(x)$?

b) Sketch a graph of $y = f(x) - g(x)$ on the interval $(-2, 2)$.

c) For what region is $f(x) - g(x) > 0$? Explain how this corresponds to your answer to part a).

4. Let $u(x) = x + 5$ and $v(x) = 2^x$.

a) Graph these functions on the same set of axes.

b) Graph the combined function $y = \dfrac{u(x)}{v(x)}$.

c) Explain how $y = \dfrac{u(x)}{v(x)}$ can be used to identify the regions where

i) $u(x) > v(x)$

ii) $v(x) < u(x)$

5. Let $f(x) = x^3 + 8x^2 - 11x - 12$ and $g(x) = -x^2 - x + 12$.

a) Graph these functions on the same set of axes.

b) Identify, by inspecting the graphs, the intervals for which

i) $f(x) > g(x)$

ii) $g(x) > f(x)$

6. Solve question 5b) using two other methods.

7. Refer to Example 1.

a) Explain how you can determine an algebraic equation for $y = f(x) - g(x)$.

b) Find this equation.

c) Could the combined function $y = g(x) - f(x)$ be used to show where $f(x) > g(x)$ and $g(x) > f(x)$? Explain.

For help with questions 8 and 9, refer to Example 2.

8. Solve $\sin x < x$.

9. Solve $x^2 > 2^x$.

B Connect and Apply

For help with questions 10 to 12, refer to Example 3.

Use the following information to answer questions 10 to 12.

The owner of a small movie theatre needs to identify the optimum price for admission tickets to maximize her profits. The number, N, of people who attend a movie showing is a function of the price, p, in dollars, $N(p) = -(p + 7)(p - 15)$, assuming a minimum ticket price of \$5.

10. a) Graph $N(p)$.

b) Identify the region for which $N(p) > 0$. What does this suggest about the maximum realistic ticket price? Explain your answer.

c) Identify the domain and range for which $y = N(p)$ has meaning.

11. **Use Technology** The revenue, R, in dollars, generated from ticket sales is given by $R(p) = N(p) \times p$, where p is the number of tickets sold.

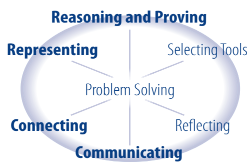
a) Graph the function $R(p)$ on a graphing calculator.

b) For what region is $R(p) > 0$? Does this result agree with the one found in question 10b)? Explain.

c) Do the maxima of $N(p)$ and $R(p)$ occur at the same value of p? Explain why or why not.

d) What does $R(p)$ suggest that the optimum ticket price is? Explain.

12. **Use Technology** The cost, C, of showing a movie can be modelled by a composite function of $N(p)$, $C(p) = 150 + 5N(p)$.



a) Graph the function $C(p)$ on a graphing calculator.

b) Graph the combined function $y = R(p) - C(p)$. What does this function represent?

c) Identify the region for which $R(p) - C(p) > 0$. What is the significance of this region?

d) Do the maxima of $y = R(p)$ and $y = R(p) - C(p)$ occur for the same value of p? Explain why or why not.

e) Identify the optimum ticket price per movie and determine the maximum profit per showing.

13. A man hanging from a spring is set into vertical motion. Its displacement, d, in centimetres, from the equilibrium point ($d = 0$) can be modelled by the function $d(t) = 5[\sin(6t) - 4\cos(6t)]$, where t is the time, in seconds. At what times during the first 2 s is the mass below the equilibrium position? Round answers to the nearest hundredth of a second.

14. Write two functions, $f(x)$ and $g(x)$, for which $f(x) > g(x)$ on the interval $(-\infty, \infty)$.

15. **Chapter Problem** It is with great fanfare that Funky Stuff announces their release of the vintage hula hoop!

The marketing department needs to establish an acceptable range of prices for the hula hoop. Based on market research, the projected revenue, R, as a function of the number, n, in thousands, of units sold can be modelled by the function $R = -0.2n^2 + 3n$. The total cost, C, as a function of n can be modelled by the function $C = 0.1n^2 + 0.4n + 2$.

a) Graph R and C on the same set of axes.

b) How many points of intersection does this system of quadratic equations have? Explain their significance.

c) Identify the region where $R > C$. Why is this region important?

d) Maximum profit occurs when R exceeds C by the greatest amount. Use the superposition principle to graph the function $P = R - C$.

e) Use this function to determine

i) the optimum number of units sold

ii) the maximum profit per unit sold

iii) the total profit, if the optimum number of units are sold

f) Reflect on the shapes of the revenue and cost curves. Suggest some reasons why they are shaped like this, instead of being linear functions.

16. Claire builds and sells birdhouses. Claire makes n birdhouses in a given week and sells them for $45 - n$ dollars per birdhouse. Her costs include a fixed cost of $280 plus $8 per birdhouse made. Assume that Claire sells all of the birdhouses that she makes.

a) Write an equation to represent her total weekly cost.

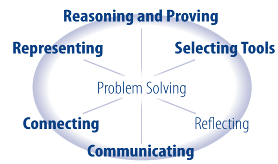
b) Write an equation to represent her total weekly revenue.

c) Write an inequality to express the conditions for which Claire will make a profit.

d) How many birdhouses should Claire build each week in order to make a profit?

e) What is the optimum number of birdhouses Claire should build in order to earn maximum profit? How much will she earn if she does this?

✓ Achievement Check



17. The projected population, P, of a town can be modelled by the function $P(t) = 2000(1.025)^t$, where t is the time, in years, from now. The expected number, N, of people who can be supplied by local services can be modelled by the function $N(t) = 5000 + 57.5t$.

a) Determine $y = N(t) - P(t)$ and sketch its graph.

b) Explain what the function in part a) represents.

c) When is $N(t) - P(t) < 0$? Explain what this means.

d) Determine $y = \dfrac{N(t)}{P(t)}$ and sketch its graph.

e) Explain what the function in part d) represents.

C Extend and Challenge

18. Write two functions, $f(x)$ and $g(x)$, for which $f(x) > g(x)$ on regular periodic intervals.

19. **a)** Write two functions, $f(x)$ and $g(x)$, for which $f(x) > g(x)$ on the interval $(0, 4)$.

b) Determine a different solution to part a).

20. $f(x) > 0$ on the interval $(-3, 3)$ and $g(x) - f(x) > 0$ on the intervals $(-\infty, -3) \cup (2, \infty)$.

a) Determine possible equations for $f(x)$ and $g(x)$.

b) Is there more than one solution for part a)? Explain.

21. $f(x)$ is a quadratic function and $g(x)$ is a linear function for which $\dfrac{f(x)}{g(x)} > 1$ on the intervals $(-\infty, -3)$ and $(4, \infty)$. Determine possible equations for $f(x)$ and $g(x)$.

22. **Math Contest** A total of 28 handshakes are exchanged at the end of a party. Assuming that everyone shakes hands with everyone else at the party, how many people are at the party?

23. **Math Contest** Determine the value of $x + y + z$, given that $x^2 + y^2 + z^2 = xy + xz + yz = 3$.

24. **Math Contest** Bill has $1.64 in pennies, nickels, dimes, and quarters. If he has equal numbers of each coin, how many coins does he have?

25. **Math Contest** Prove that given any three consecutive numbers, one of these numbers will always be divisible by three.

8.5 Making Connections: Modelling With Combined Functions

Have you ever bungee jumped? If you were to bungee jump, how many bounces do you suppose you would make? When would you be travelling the fastest?

Many of these questions and others like it can be answered by mathematical modelling with combined functions. Are you ready to take the leap?

Investigate — How can a combined function be used to model the path of a bungee jumper?

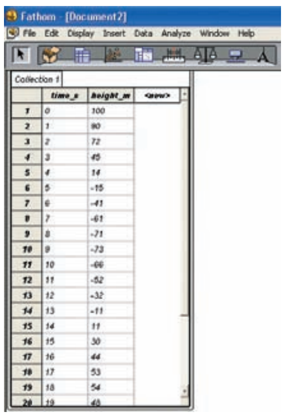
Tools

- computer with *Fathom*™

The height versus time data of a bungee jumper are given in the table, for the first half minute or so of his jump. Heights are referenced to the rest position of the bungee jumper, which is well above ground level.

Time (s)	0	1	2	3	4	5	6	7	8	9	10	11	12
Height (m)	100	90	72	45	14	−15	−41	−61	−71	−73	−66	−52	−32
Time (s)	13	14	15	16	17	18	19	20	21	22	23	24	25
Height (m)	−11	11	30	44	53	54	48	37	23	6	−8	−23	−33
Time (s)	26	27	28	29	30	31	32	33	34	35	36	37	
Height (m)	−39	−39	−35	−27	−16	−4	6	17	24	28	29	26	

1. In *Fathom*™, open a new collection and enter the data in a case table.



2. a) Create a scatter plot of height versus time.

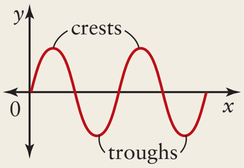
b) Describe the shape formed by the points.

3. a) Describe how the graph exhibits sinusoidal features.

b) Describe how the graph exhibits exponential features.

CONNECTIONS

The *crests* of a wave are its highest points, or local maximum values. The *troughs* are its lowest points, or local minimum values.



Technology Tip

To enter one of the attributes in a case table into a calculation, you can either type it in manually or select it from the **attributes** drop-down menu.

You can manipulate the scale of a slider by placing the cursor over different regions of the slider. Experiment to see how you can

- slide the scale
- stretch the scale
- compress the scale

4. a) Construct a cosine function that has the same period (or wavelength) as the scatter plot by following these steps:

- Create a slider and call it k.
- Click on the graph. From the **Graph** menu, choose **Plot Function**.
- Enter the function 100*cos(k*time_s).
- Adjust the slider until the crests and troughs of the function occur at the same times as for the scatter plot.

b) Note the value of k.

5. a) Construct an exponential function to model the decay in amplitude of the scatter plot by following these steps:

- Create a slider and call it c.
- Click on the graph. From the **Graph** menu, choose **Plot Function**.
- Enter the function 100*0.5^(c*time_s).
- Adjust the slider until the exponential curve just touches each crest of the scatter plot.

b) Note the value of c.

6. Construct a combined function to model the height-time relationship of the bungee jumper by following these steps:

- Click on the graph. From the **Graph** menu, choose **Plot Function**.
- Enter the function 100*cos(k*time_s)*0.5^(c*time_s).

Make any final adjustments to k or c, if necessary.

7. a) How far will the bungee jumper be above the rest position at his fourth crest?

b) For how long will the bungee jumper bounce before his amplitude has diminished to 10 m?

8. Reflect

a) What is the significance of the factor of 100?

b) Why was a cosine function chosen instead of a sine function?

c) When the cosine function and the exponential function are combined, only one factor of 100 is used. Why?

d) When is the magnitude of the rate of change of this function the greatest? What does this mean from a physical perspective?

Example 1 Combining Musical Notes

Musical notes are distinguished by their pitch, which corresponds to the frequency of vibration of a moving part of an instrument, such as a string on a guitar. The table lists one octave of the frequencies of commonly used notes in North American music, rounded to the nearest hertz (Hz). This is called the chromatic scale.

Note	Frequency (Hz)	Note	Frequency (Hz)
C	262	F#	370
C#	277	G	392
D	294	G#	415
D#	311	A	440
E	330	A#	466
F	349	B	494
---	---	high C	524

The graph of a pure note can be modelled by the function $I(t) = \sin(2\pi ft)$, where I is the sound intensity; f is the frequency of the note, in hertz; and t is the time, in seconds.

According to music theory, notes sound good together when they cause constructive interference at regular intervals of time.

a) Graph the intensity functions for C and high C. Then, graph the combined function for these two notes struck together. Describe the resultant waveform.

b) A C-major triad is formed by striking the following notes simultaneously:

C E G

Graph the combined function for these notes struck together. Explain why these notes sound good together.

c) Graph the intensity functions for C and F# (F-sharp) and the combined function for these two notes struck together. Explain why these notes are discordant (do not sound good together).

Solution

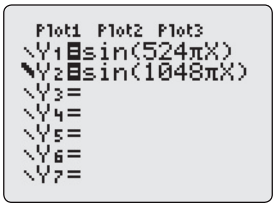
a) Determine the waveforms for C and high C by substituting their frequencies into the intensity function.

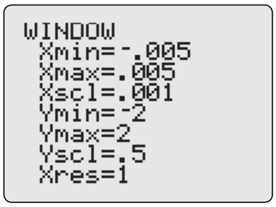
$$I_C(t) = \sin[2\pi(262)t] = \sin(524\pi t)$$

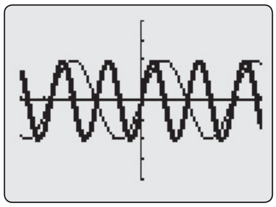
$$I_{\text{high C}}(t) = \sin[2\pi(524)t] = \sin(1048\pi t)$$

Use a graphing calculator to graph these waveforms.

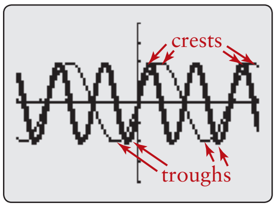
Ensure that the calculator is set to Radian mode. It may be necessary to experiment with the window settings to get a clear view of the waveforms.



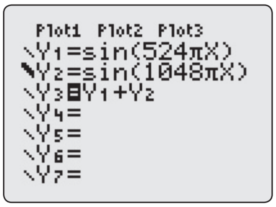


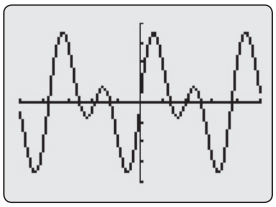


Note that high C has the same waveform as C, compressed by a horizontal factor of $\frac{1}{2}$, resulting in regular occurrences of constructive interference (their crests and troughs occur at nearly the same points).



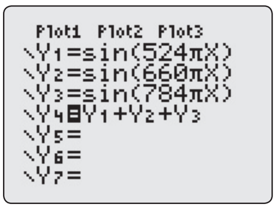
This suggests that these notes will sound good together. To view the combination of these two waveforms, apply the superposition principle. Turn off the functions for the two component waveforms for clarity.

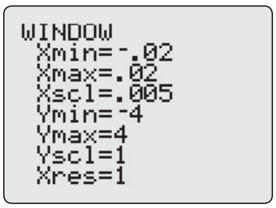


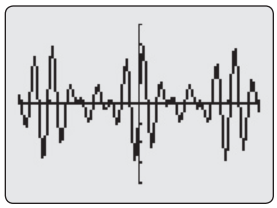


This waveform is periodic. It is basically sinusoidal with a slight variation at regular intervals. These notes will sound good together.

b) To graph the C-major triad, enter the waveforms for the C, E, and G notes, and then graph their sum using the superposition principle. Use the frequencies given in the table.

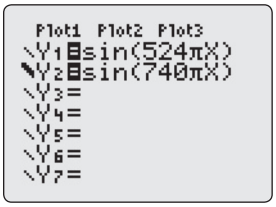


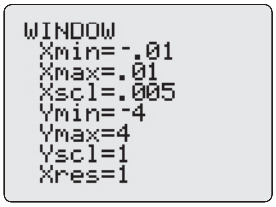


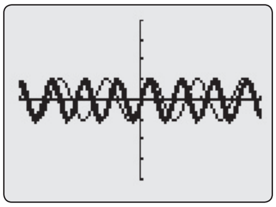


This waveform is considerably more complicated than a simple sinusoidal waveform. Note, however, that there is still a regular repeating pattern to the wave, which suggests that these notes will also sound good together. The multiple variations along the cycle actually give the chord its interesting character.

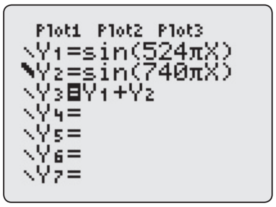
c) Use graphing technology to view the waveforms for the notes C and F#.

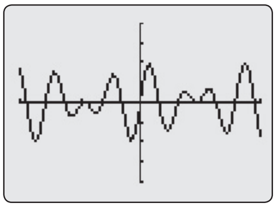






Notice that the crests and troughs of these two notes do not coincide at regular intervals, suggesting that these two notes will not sound good together. View the graph of their sum using the superposition principle to confirm this.

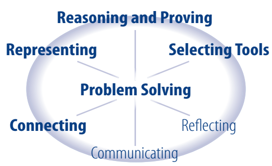




This waveform does not appear to be periodic, confirming that C and F# are discordant notes.

Example 2 Path of a Skier

A skier is skiing down a 100-m hill at a constant speed of 1 m/s, through a series of moguls, or small hills. The constant slope of the hill is -1. Assuming that the moguls measure 1.5 m from crest to trough and are roughly 5 m apart, develop an algebraic and a graphical model of the height of the skier versus time.

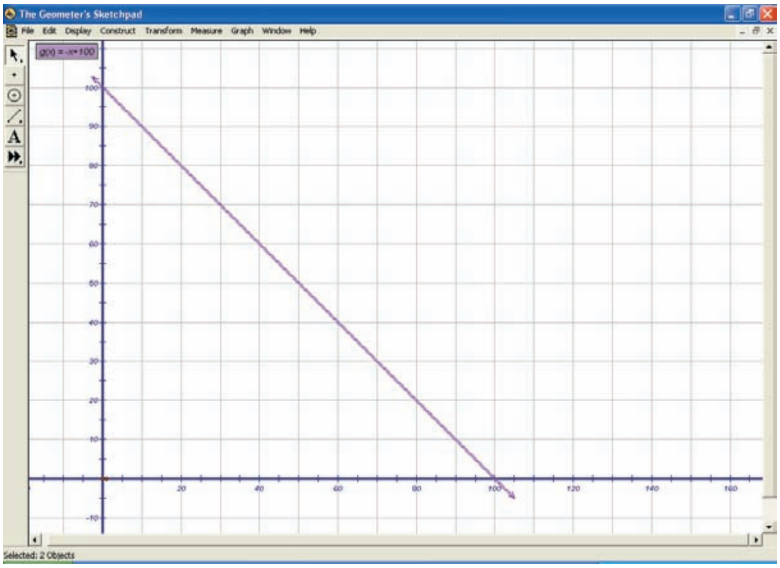


Solution

Break the desired function into two parts. First, determine a function for the skier's height assuming that there are no moguls. Then, add the effect of the moguls.

Neglecting the effect of the moguls, the path of the skier, as a function of height versus time, can be modelled by a straight line.

The hill is 100 m high, and the rate of change of height versus time is -1 m/s. Therefore, the height of the skier can be approximated by the function $h(t) = -t + 100$. Using *The Geometer's Sketchpad*®, graph this function.

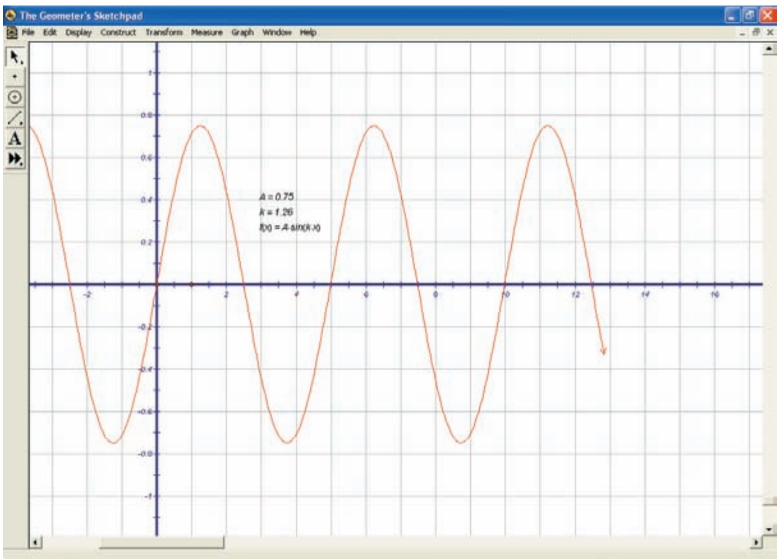


To model the skier's height through the moguls, ignoring the slope of the hill for the moment, generate a sinusoidal function to reflect the correct amplitude and wavelength.

Method 1: *Use Parameters*

Using *The Geometer's Sketchpad*®, create two parameters to represent the amplitude, A, and the horizontal compression, k. Then, construct the function $f(x) = A\sin(kx)$.

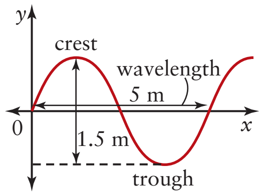
Adjust the parameters until the amplitude is $1.5 \div 2$, or 0.75 m, and the wavelength is 5 m.



Therefore, the height, M, in metres, of the skier through the moguls, ignoring the slope of the hill, after t seconds, is $M(t) = 0.75\sin(1.26t)$.

Method 2: *Apply Algebraic Reasoning*

The crest-to-trough distance of the moguls is 1.5 m, and they are 5 m apart.



The amplitude, A, is half the crest-to-trough distance:

$$A = 1.5 \div 2$$
$$= 0.75$$

The average speed of the skier is 1 m/s, so the frequency, f, is $\frac{1}{5}$, or 0.2 moguls per second.

Substitute these values for A and f into the function $M(t) = A\sin(2\pi ft)$, where M represents the height, in metres, within the mogul as a function of time, t, in seconds.

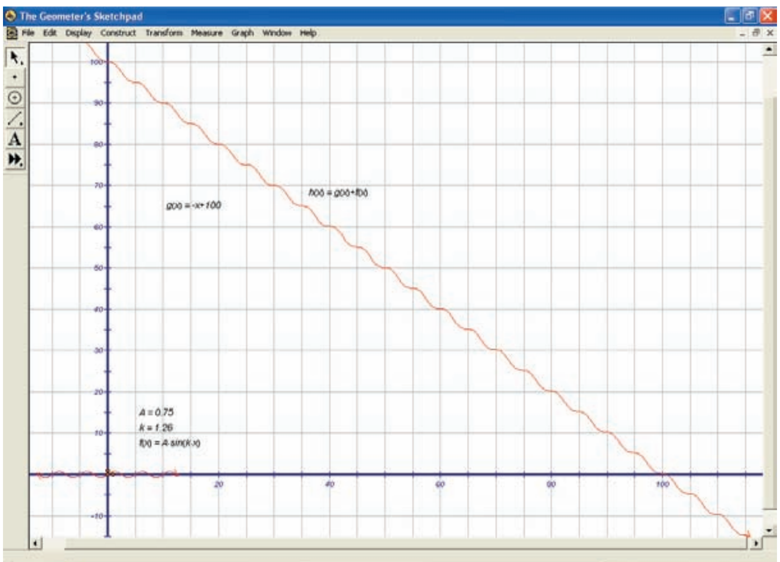
$$M(t) = 0.75\sin(2\pi(0.2)t)$$
$$= 0.75\sin(0.4\pi t)$$

Therefore, the height of the skier through the moguls, ignoring the slope of the hill, is $M(t) = 0.75\sin(0.4\pi t)$, or $M(t) \doteq 0.75\sin(1.26t)$.

To describe the skier's actual path of height versus time, add the two functions.

$$P(t) = h(t) + M(t)$$
$$= -t + 100 + 0.75\sin(1.26t)$$

The graph of this function is shown.



CONNECTIONS

Jennifer Heil became the first female Canadian Olympic gold medal winner in mogul skiing in the freestyle moguls event at the 2006 Winter Games in Turin, Italy.

This Canadian hero, who is also a champion ski racer, then went on to become the 2007 CanWest athlete of the year.

KEY CONCEPTS

- A variety of real-world situations can be modelled using combined functions.
- To develop a model consisting of a combined function, consider
  - the component functions that could be combined to form the model
  - the nature of the rate of change of the component functions
  - the other key features of the graph or equation that fit the given scenario

Communicate Your Understanding

C1 Refer to the Investigate. A simple harmonic oscillator is an object that repeats a cyclic motion indefinitely, such as a pendulum on a clock. A damped harmonic oscillator is an object that repeats its cyclic motion, but whose amplitude decreases over time due to frictional forces.

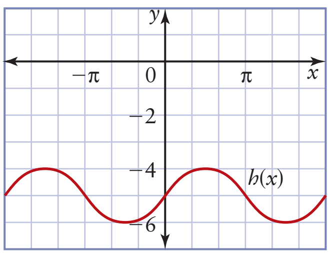
a) What is the difference between simple harmonic motion and damped harmonic motion?

b) Is the motion of the bungee jumper an example of damped harmonic motion? Explain why or why not.

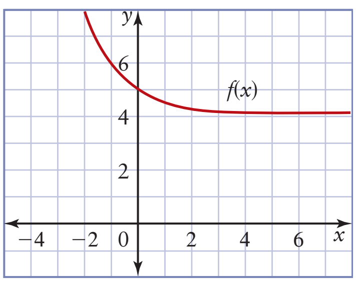
c) Describe two other scenarios that are examples of damped harmonic motion.

C2 Each graph is made up of a sum or difference of two functions. Identify the types of functions that are combined in each case.

a)



b)

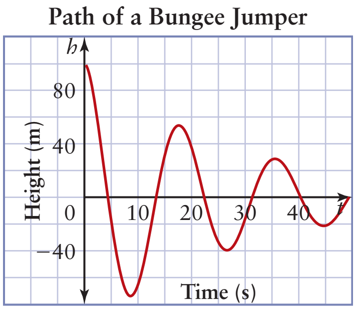


C3 Refer to question C2. Suggest a possible scenario that could be modelled by each graph.

A Practise

For help with questions 1 to 3, refer to the Investigate.

Use this graph of the path of the bungee jumper to answer questions 1 to 3.



1. Copy the graph of this curve. Sketch how the path of the jumper would change if she dropped from

a) a greater height

b) a lower height

2. Sketch how the path of the jumper would change if she were attached to

a) a longer cord

b) a shorter cord

3. Sketch how the path of the jumper would change if she were attached to

a) a springier cord

b) a stiffer cord

For help with questions 4 to 6, refer to Example 1.

4. A C-major triad can be expanded into other C-major chords by adding additional notes of the triad from the next octave.

a) Graph the combined function formed by the following notes being struck simultaneously:

C E G high C high E

b) Compare this waveform to the one for the C-major triad.

5. A power chord is formed by dropping the major third (or E-note) from the C-major triad.

a) Graph each C power chord.

i) C G

ii) C G high C

b) Compare these waveforms to those of the C-major triad and the C-major chord.

CONNECTIONS

Power chords are widely used in heavy metal music. In addition to generating a full, powerful sound, they are relatively easy to fret on a guitar, making it possible for even an amateur player to work quickly along the fret board to generate a blistering rhythm.

6. Bobby attempts to play a C-major triad on his guitar, but it is badly tuned. The C-note is correctly tuned to 262 Hz, but the E and G notes are flat (lower in frequency than they should be), as follows:

E: 300 Hz
G: 360 Hz

a) Graph the C-major triad as played by Bobby's guitar.

b) Discuss the sound quality of this chord, making reference to the waveform.

For help with questions 7 to 9, refer to Example 2.

7. Refer to Example 2. Sketch how the graph would change if the moguls were

a) higher

b) closer together

8. A skier is going up and down the same hill at regular intervals. On each run, she skis at an average speed of 1 m/s from the top of the hill to the bottom, waits in the chairlift line for about 1 min, and then travels up the chairlift at a speed of 0.5 m/s. Assume that this hill has no moguls.

a) Develop a graphical model of the skier's height versus time over the course of several ski runs.

b) Explain what is happening during each region of the graph for one cycle.

c) How might you develop an algebraic model to describe this motion?

9. Refer to question 8. Adjust your graph from part a) to illustrate the effect on the skier's height function in each scenario.

a) After her second run, she meets a friend and spends an extra minute chatting in the lift line.

b) On her third trip up the chairlift, the lift is stopped for 30 s so that other skiers can assist someone who is having trouble getting on the lift.

c) On her third trip down the hill, the skier doubles her speed.

B Connect and Apply

The number of teams in the National Hockey League (NHL) is given for the past several decades.

Year	Teams	Year	Teams	Year	Teams
1966	6	1980	21	1994	26
1967	12	1981	21	1995	26
1968	12	1982	21	1996	26
1969	12	1983	21	1997	26
1970	14	1984	21	1998	27
1971	14	1985	21	1999	28
1972	16	1986	21	2000	30
1973	16	1987	21	2001	30
1974	18	1988	21	2002	30
1975	18	1989	21	2003	30
1976	18	1990	21	2004	30
1977	18	1991	22	2005	30
1978	17	1992	24	2006	30
1979	21	1993	26	2007	30

Use this information to answer questions 10 to 12.

10. a) Using 1966 as year 0, create a scatter plot of the number, N, of teams versus time, t, in years.

b) Determine a line or curve of best fit, using a method of your choice (e.g., regression analysis, sliders, systematic trial). Justify the type of function that you chose.

c) Write an equation for $N(t)$.

d) There is an outlier in 1966 that does not appear to fit the trend very well. What effect does removing the outlier have on the model for $N(t)$? Does this effect appear to be significant?

CONNECTIONS

You will study the effect of outliers on lines and curves of best fit if you study data management.

Use a line or curve of best fit for N(t) to answer questions 11 and 12.

11. According to one hockey analyst, the number of high-calibre players emerging from the junior ranks can be modelled by the function $D(t) = 25 + 0.4t$, where D is the number of drafted players who are considered talented enough to play at the NHL level as a function of time, t, in years.

a) Graph this function and describe the trend. What does this suggest about the junior leagues?

b) The number, R, of players retiring each year is a function of the number of teams and time given by $R(t) = 2[N(t)] - 0.53t$. Graph $R(t)$ on the same set of axes as in part a) and describe the trend.

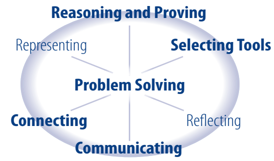
c) What does this suggest about the length of NHL players' careers?

d) Graph the function $y = \frac{R(t)}{N(t)}$. Does this confirm your answer to part c)? Explain why or why not.

e) Graph the function $P(t) = D(t) - R(t)$. What does this function describe?

f) Describe the trend of $P(t)$, and discuss any implications it might have.

12. a) On a new set of axes, graph the function $y = \frac{P(t)}{N(t)}$. What does this function describe?



b) What is the significance of the t-intercept? What implications might this have for the quality of NHL hockey?

c) Suppose that the function $D(t)$ is adjusted to recognize the increase in draftable players from foreign countries, such that the slope increases slightly. Discuss the effect this will have on the functions $y = P(t)$ and $y = \frac{P(t)}{N(t)}$. How might this affect your answers to part b)?

13. Use Technology

a) Using a motion probe and a graphing calculator, capture the motion of a person on a swing as he or she moves back and forth relative to the sensor, while undergoing damped harmonic motion.

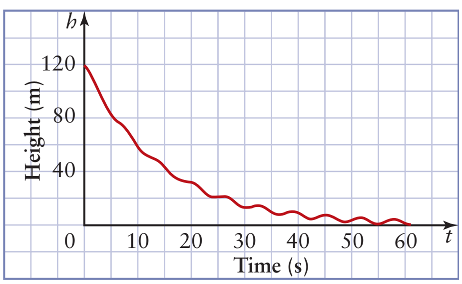
b) Create a scatter plot of the data.

c) Develop an equation to model the data. Explain how you determined the equation.

C Extend and Challenge

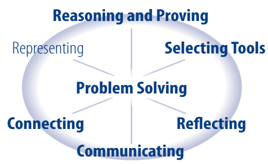
14. A skier is skiing down a hill at a constant speed. Her height versus time graph is shown.

Path of a Skier



Develop an algebraic model for this relationship.

15. Perform research on the Internet or using other sources. Find an example of a real-world situation in which a combined function is, or could be, used to model a relationship. Explain



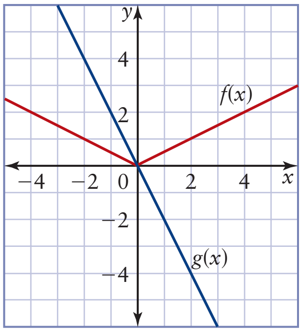
- the nature of the relationship being described
- the types of component functions that are used
- the way the functions are combined

Pose and solve two problems based on your research.

Chapter 8 REVIEW

8.1 Sums and Differences of Functions

1. a) Copy the graph.



b) Use the superposition principle to generate a graph of each function.

i) $y = f(x) + g(x)$

ii) $y = f(x) - g(x)$

iii) $y = g(x) - f(x)$

2. Let $f(x) = x - 2$, $g(x) = x^2 + 3x - 3$, and $h(x) = 2^x$. Determine an algebraic and a graphical model for each combined function. Identify the domain and range in each case.

a) $y = f(x) + g(x)$

b) $y = f(x) + g(x) + h(x)$

c) $y = f(x) - h(x)$

3. Use Technology Use graphing technology to check your answers to question 2.

4. Max can earn \$6/h as a waiter, plus an additional \$9/h in tips.

a) Graph Max's earnings from wages as a function of hours worked.

b) Graph Max's earnings from tips as a function of hours worked.

c) Develop an algebraic and a graphical model for Max's total earnings.

d) How much can Max earn if he works 52 h in one week?

8.2 Products and Quotients of Functions

5. Let $u(x) = x^2$ and $v(x) = \cos x$. Work in radians.

a) What type of symmetry do you predict the combined function $y = u(x)v(x)$ will have? Explain your reasoning.

b) Use Technology Use graphing technology to check your prediction.

6. Let $f(x) = \sin x$ and $g(x) = \cos x$.

a) Graph $f(x)$ and $g(x)$ on the same set of axes.

b) Sketch a graph of the combined function $y = \frac{f(x)}{g(x)}$.

c) Identify the domain and range of this function.

d) Use your understanding of trigonometric identities to identify the graph of $y = \frac{f(x)}{g(x)}$.

7. Refer to question 6.

a) Graph the combined function $y = \frac{g(x)}{f(x)}$.

b) Identify the domain and range of this function.

c) How is $y = \frac{g(x)}{f(x)}$ related to $y = \frac{f(x)}{g(x)}$ in terms of transformations?

8.3 Composite Functions

8. Let $f(x) = x^2 + 3x$ and $g(x) = 2x - 5$. Determine an equation for each composite function, graph the function, and give its domain and range.

a) $y = f(g(x))$

b) $y = g(f(x))$

c) $y = g(g(x))$

d) $y = g^{-1}(g(x))$

9. Assume that a function $f(x)$ and its inverse $f^{-1}(x)$ are both defined for $x \in \mathbb{R}$.

a) Give a geometric interpretation of the composite function $y = f(f^{-1}(x))$.

b) Illustrate your answer to part a) with two examples.

8.4 Inequalities of Combined Functions

10. Let $f(x) = 1.2^x$ and $g(x) = 0.92^x + 5$.

a) Identify the region for which

i) $f(x) > g(x)$

ii) $g(x) > f(x)$

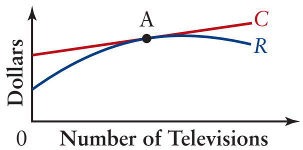
b) Illustrate this inequality graphically in two different ways.

11. Refer to question 10.

a) Write a real-world scenario that these functions could model.

b) Pose and solve two problems based on your scenario.

12. The cost, C, and revenue, R, as functions of the number of televisions sold by an electronics store are shown on the graph.



a) Identify the region(s) for which

i) $C > R$

ii) $R > C$

b) What can you conclude about this business venture?

c) What suggestions would you give to the store owner in order to help him or her improve the situation?

8.5 Making Connections: Modelling with Combined Functions

Refer to the chromatic music scale on page 463.

13. A D-minor chord is formed by striking the following notes together:

D F A high D

a) Double the frequency of D in the table to determine the frequency of high D.

b) Graph the combined function formed by these four notes. Describe the waveform.

CONNECTIONS

Minor chords tend to have a sad sound to them. They combine with major chords (which sound happier) to create musical tension.

14. Experiment with various note combinations from the chromatic scale.

a) Identify two chords that you think would make a good sound.

b) Identify two chords that you think would make a discordant (unpleasant) sound.

c) Use mathematical reasoning to justify your choices. Then, test your theories using a well-tuned guitar or piano. You may need to do a little research to identify the correct notes.

CHAPTER PROBLEM WRAP-UP

The number, S, in thousands, of Funky Teddy Bears that can be supplied by Funky Stuff as a function of price, p, in dollars, can be modelled by the function $S(p) = p + 3$.

The demand, D, for the bears can be modelled by the function $D(p) = -0.1(p + 8)(p - 12)$.

a) For what interval is $D(p) > S(p)$? What does this imply about the availability of Funky Teddy Bears?

b) For what interval is $D(p) < S(p)$? What does this imply about the availability of Funky Teddy Bears?

c) Graph these functions on the same set of axes. Identify their point of intersection. Explain what the coordinates of this point mean.

d) Graph the function $y = S(p) - D(p)$ and explain what it shows.

Chapter 8 PRACTICE TEST

For questions 1 to 4, choose the best answer.

1. Which function represents the composite function $y = f(g(x))$, where $f(x) = x + 2$ and $g(x) = x^2$?

A $y = x^2 + 2$

B $y = x^2 + 2x$

C $y = x^2 + 4$

D $y = x^2 + 4x + 4$

2. Let $u(x) = x - 1$ and $v(x) = x^2 - 1$. For which values of x is the combined function $y = \frac{u(x)}{v(x)}$ undefined?

A $x \in \mathbb{R}$

B $x = 1$

C $x = -1$

D $x = \pm 1$

3. Let $f(x) = (x - 2)^2$ and $g(x) = -x + 3$. Which function represents the combined function $y = f(x) + g(x)$?

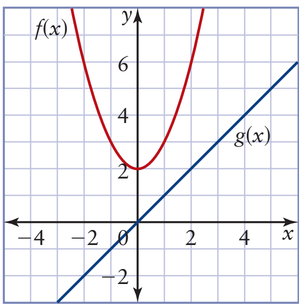
A $y = x^2 - x + 1$

B $y = -x^2 + 3x - 7$

C $y = x^2 + 3x + 7$

D $y = x^2 - 5x + 7$

4. The graphs of two functions are shown.



Which is true for $x \in \mathbb{R}$?

A $g(x) - f(x) < 0$

B $\frac{f(x)}{g(x)} > 1, x \neq 0$

C $f(x) < g(x)$

D All of the above.

5. Let $f(x) = (x - 3)^2$ and $g(x) = x + 4$. Determine an algebraic and a graphical model for each combined function. Give the domain and range in each case.

a) $y = f(x) + g(x)$

b) $y = f(x) - g(x)$

c) $y = f(g(x))$

d) $y = g^{-1}(g(x))$

6. **Use Technology** Check your answers to question 5 using graphing technology.

7. a) Create a growing pattern of tiles to represent the sum of a linear function, a quadratic function, and another function of your choice. Draw several stages of the pattern.

b) Develop an algebraic model to represent the total number of tiles required for the nth stage.

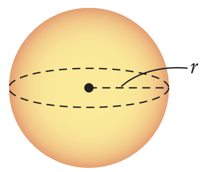
c) Draw a separate graph for each part of the growing pattern as a function of n and a graph that represents the total number of tiles required for the nth stage.

8. Let $f(x) = x + 5$ and $g(x) = x^2 + 9x + 20$.

a) Determine an algebraic and a graphical model for $y = \frac{f(x)}{g(x)}$ and identify its domain and range.

b) Determine an algebraic and a graphical model for $y = \frac{g(x)}{f(x)}$ and identify its domain and range.

9. Express the volume of a sphere of radius r as a function of



a) the circumference

b) the surface area

10. Is the difference of two odd functions odd or even? Justify your answer with two examples.

11. A pendulum is released and allowed to swing back and forth according to the equation $x(t) = 10\cos(2t) \times 0.95^t$, where x is the horizontal displacement from the rest position, in centimetres, as a function of time, t, in seconds.

a) Graph the function. What type of motion is this? Identify the domain and range in the context of this problem.

b) This combined function is the product of two component functions. Identify the component that is responsible for

i) the periodic nature of the motion

ii) the exponential decay of the amplitude

c) At what horizontal distance from the rest position was the bob of the pendulum released?

d) At what point on the graph is the magnitude of the rate of change the greatest? When does this occur with respect to the motion of the pendulum bob?

e) At what point(s) is the rate of change zero? When does this occur with respect to the motion of the pendulum bob?

f) After what elapsed time will the pendulum's amplitude diminish to 50% of its initial value?

12. A catenary is the shape of a hanging flexible cable or chain between two supports that is acted upon only by gravity. It looks like a parabola, but it is not. In reality, it is an exponential function, not a quadratic function. Once a load is applied, such as with a suspension bridge, the curve becomes parabolic.

a) Sketch a graph of the catenary defined by $f(x) = \frac{1}{2}(e^x + e^{-x})$, where $e \doteq 2.718$, on the interval $x \in [-5, 5]$.

b) Develop a quadratic function to closely model this catenary.

c) Sketch both functions, labelling common points and showing where the functions diverge.

13. The number, S, in hundreds, of swimmers at Boulder Beach as a function of temperature, T, in degrees Celsius, can be modelled by the function $S(T) = -0.05(T - 28)(T - 34)$.

The number, I, of ice-cream sales made by the Boulder Beach ice-cream vendor can be modelled by the function $I(T) = 0.5S(T) \times T$.

a) Graph $S(T)$ and $I(T)$ on the same set of axes and describe their trends.

b) At what temperature will Boulder Beach attract the greatest number of swimmers? How many will come for a cooling dip?

c) At what temperature will the Boulder Beach ice-cream vendor earn maximum profits? Is this the same temperature as the maximum found in part b)? Why or why not?

14. a) Determine two functions, $f(x)$ and $g(x)$, for which $f(x) < g(x)$ on the interval $(-1, 1)$ and $f(x) > g(x)$ on the intervals $(-\infty, -1) \cup (1, \infty)$.

b) Illustrate graphically how your functions satisfy these criteria, using three different methods. Explain each method.

15. A skier's height, h, in metres, as a function of time, t, in seconds, can be modelled by the combined function $h(t) = [80(0.9)^t + 0.5] + 0.6\sin(3t)$.

a) Graph this function.

b) The skier encounters moguls at some point during her run. Assuming that she travels at a constant speed, at approximately what point in time does she encounter the moguls? Explain how you can tell.

c) Assuming that the skier stops the first time her height reaches zero, find the domain and range relevant to this problem.

Chapters 6 to 8 REVIEW

Chapter 6 Exponential and Logarithmic Functions

1. a) Copy and complete the table of values for the function $y = 4^x$.

x	y
-2	
-1	
0	
$\frac{1}{2}$	
1	
2	
3	

b) Graph the function.

c) Sketch a graph of the inverse of $y = 4^x$.

2. Evaluate each logarithm.

a) $\log_5 25$ **b)** $\log_2 128$

c) $\log_3\left(\frac{1}{27}\right)$ **d)** $\log 1000$

e) $\log_7 7^4$ **f)** $\log 10^{-5}$

3. The intensity, I, of sunlight decreases exponentially with depth, d, below the surface of the ocean. The relationship is given by $I(d) = 100(0.39)^d$.

a) When the intensity of sunlight at the surface is 100 units, what is the intensity at a depth of 3 m, to the nearest tenth?

b) What is the intensity at a depth of 5 m, to the nearest tenth?

4. a) Graph the function $y = \log(2x - 4)$. Determine the domain, the range, the x- and y-intercepts, and the equations of any asymptotes.

b) Describe how a graph of the function $y = -\log(2x - 4)$ would differ from the graph in part a).

5. Evaluate.

a) $\log_6 36^5$

b) $\log_4 32 + \log_4 2$

6. Solve for x. Round answers to two decimal places, if necessary.

a) $4^x = 15$

b) $x = \log_2 18$

7. A vehicle depreciates by 25% each year. Its value, V, in dollars, as a function of time, t, in years, can be modelled by the function $V(t) = 28\,000(0.75)^t$.

a) What was the initial value of the vehicle? Explain how you know.

b) How long will it take for the vehicle to depreciate to half its initial value, to the nearest tenth of a year?

8. The decibel rating of a sound is $\beta = 10\log\frac{I}{I_0}$, where $I_0 = 10^{-12}$ W/m^2.

a) Find the decibel rating of a rock concert with an intensity of 8.75×10^{-3} W/m^2.

b) A radio on very low, so it is just audible, has a decibel rating of 20 dB. What is the intensity of the sound from this radio?

Chapter 7 Tools and Strategies for Solving Exponential and Logarithmic Equations

9. Solve. Check your answers using graphing technology.

a) $27^{x+2} = 9^{5-2x}$

b) $10^{5x+4} = 1000^{3x}$

c) $64^{x+5} = 16^{2x-1}$

10. Solve for n. Round answers to two decimal places.

a) $8 = 3^n$ **b)** $5.8^n = 100$

c) $10^{n+5} = 7$ **d)** $2^{-n} = 6$

e) $278^{3n-7} = 21^{2n+5}$ **f)** $5^{2n} = 0.75^{n-4}$

11. Various radioactive substances are used in medical tests.

a) Iodine-131 is used in thyroid tests. A 20-mg sample decays to 16.85 mg in 48 h. What is the half-life of iodine-131, to the nearest hour?

b) Strontium-87 is used in some bone tests. After 1 h, a 100-mg sample decays to 78.1 mg. Determine the half-life of strontium-87, to the nearest tenth of an hour.

12. Simplify. State any restrictions on the variables.

a) $\log(x^2 - 1) - \log(x - 1)$

b) $\log\sqrt{x} + \log x^3 - 2\log x$

c) $\log\left(1 + \frac{2y}{x} + \frac{y^2}{x^2}\right) + 2\log x$

13. Solve. Identify and reject any extraneous roots. Check your solutions using graphing technology.

a) $\log_3(-3x + 5) = 2$

b) $\log(x + 5) - \log(x + 1) = \log(3x)$

c) $\log_5(x - 6) = 1 - \log_5(x - 2)$

14. According to the power law of logarithms, $\log x^2 = 2\log x$. Are the graphs of the functions $f(x) = \log x^2$ and $g(x) = 2\log x$ the same? Explain any differences.

15. The population of a city is increasing by 12% annually. The population is 38 000 now.

a) Write an equation to show the population as a function of time, t, in years.

b) How long will it take for the population to double, to the nearest tenth of a year?

c) After how many years will the population reach 100 000?

Chapter 8 Combining Functions

16. Given $f(x) = 2^x + 2$, $g(x) = x^2 - 1$, and $h(x) = 2x$, determine an algebraic and a graphical model for each function. Identify the domain and the range for each.

a) $y = f(x) + g(x)$

b) $y = f(x) - g(x) - h(x)$

c) $y = f(x)g(x)$

d) $y = \frac{f(x)}{g(x)}$

17. Kathy has a small business selling apple cider at the farmers' market. She pays $35 per day to rent her space at the market, and each cup of cider costs her $1. She sells the cider for $2.50 per cup and brings enough cider and cups to sell a maximum of 200 cups in a day.

a) Write an equation to model each of the following for 1 day, as a function of the number, n, of cups of cider sold.

i) her total cost, C

ii) her revenue, R

b) Graph $C(n)$ and $R(n)$ on the same set of axes.

c) Identify the break-even point and explain what its coordinates mean.

d) Develop an algebraic and a graphical model for the profit function, $P(n)$.

e) What is the maximum daily profit Kathy can earn?

18. Consider $f(x) = x + 3$ and $g(x) = \cos x$, where x is in radians.

a) Describe and sketch a graph of $f(x)$. Is this function even, odd, or neither?

b) Describe and sketch a graph of $g(x)$. Is this function even, odd, or neither?

c) Predict the shape of $y = f(x)g(x)$. Sketch a graph of your prediction.

d) **Use Technology** Check your work by graphing the three functions in parts a) to c) using graphing technology.

e) Give the domain and the range of $y = f(x)g(x)$.

19. Let $f(x) = \sqrt{x - 9}$ and $g(x) = \frac{1}{x^2}$. Write a simplified algebraic model for each composite function. State the domain and the range of each.

a) $y = f(g(x))$

b) $y = g(f(x))$

20. Let $f(x) = x^3 - 3x^2 - 40x$ and $g(x) = 25 - x^2$.

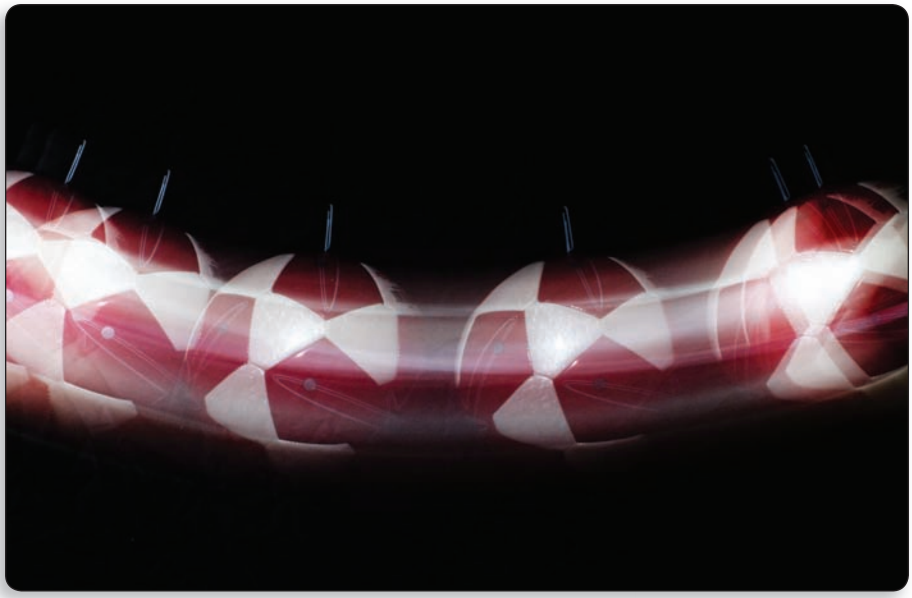
a) Graph these functions on the same set of axes.

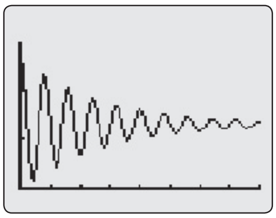
b) Identify, by inspecting the graphs, the interval(s) for which $f(x) > g(x)$.

c) Check your answer to part b) using another method.

TASK

Modelling a Damped Pendulum





Jasmeet built a simple pendulum by fastening a soccer ball to a piece of string. She then swung the pendulum in front of a motion detector and obtained the graph shown on her graphing calculator. The horizontal scale is in 1-s intervals, and the vertical scale is in 1-m intervals.

a) Compare this graph to functions you have studied. Describe the similarities and the differences.

b) Use a trigonometric function to model the graph, with or without using technology. Match the initial amplitude and period. You will model the decreasing amplitude in the next two steps.

c) Locate the maximum values for the function. Construct a table of values for these maximum values. Subtract the vertical displacement from these values. Then, determine an exponential model for the maximum values, with or without using technology.

d) Use your model from step c) as part of the amplitude for your model from step b), and graph this blended model using technology. Add the appropriate vertical displacement. Discuss how well your model matches the real situation.

e) What physical factors explain the shape of the original graph? What other situations in the real world might give rise to a similar graph?

Chapter 1 Polynomial Functions

1. a) What is meant by even symmetry? odd symmetry?

b) Describe how to determine whether a function has even or odd symmetry.

2. Describe two key differences between polynomial functions and non-polynomial functions.

3. Compare the end behaviour of the following functions. Explain any differences.

$f(x) = -3x^2 \qquad g(x) = 5x^4 \qquad h(x) = 0.5x^3$

4. Determine the degree of the polynomial function modelling the following data.

x	y
−2	17
−1	−3
0	−3
1	−1
2	33
3	177

5. Determine an equation and sketch a graph of the function with a base function of $f(x) = x^4$ that has been transformed by $-2f(x - 3) + 1$.

6. Sketch the functions $f(x) = x^3$ and $g(x) = -\frac{1}{2}(x - 1)(x + 2)^2$ on the same set of axes. Label the x- and y-intercepts. State the domain and range of each function.

7. Consider the function $f(x) = 2x^4 + 5x^3 - x^2 - 3x + 1$.

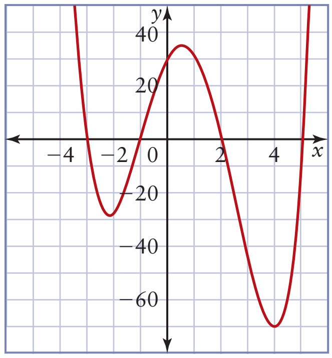
a) Determine the average slope between the points where $x = 1$ and $x = 3$.

b) Determine the instantaneous slope at each of these points.

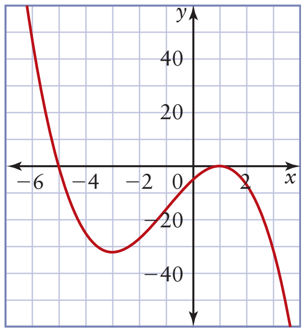
c) Compare the three slopes and describe how the graph is changing.

8. Determine an equation for each function.

a)



b)



9. Given the function $f(x) = -2x^2 + 1$, describe the slope and the change in slope for the appropriate intervals.

Chapter 2 Polynomial Equations and Inequalities

10. Perform each division. Write the statement that can be used to check each division. State the restrictions.

a) $(4x^3 + 6x^2 - 4x + 2) \div (2x - 1)$

b) $(2x^3 - 4x + 8) \div (x - 2)$

c) $(x^3 - 3x^2 + 5x - 4) \div (x + 2)$

d) $(5x^4 - 3x^3 + 2x^2 + 4x - 6) \div (x + 1)$

11. Factor, if possible.

a) $x^3 + 4x^2 + x - 6$

b) $2x^3 + x^2 - 16x - 15$

c) $x^3 - 7x^2 + 11x - 2$

d) $x^4 + x^2 + 1$

12. Use the remainder theorem to determine the remainder for each.

a) $4x^3 - 7x^2 + 3x + 5$ divided by $x - 5$

b) $6x^4 + 7x^2 - 2x - 4$ divided by $3x + 2$

13. Use the factor theorem to determine whether the second polynomial is a factor of the first.

a) $3x^5 - 4x^3 - 4x^2 + 15; x + 5$

b) $2x^3 - 4x^2 + 6x + 5; x + 1$

14. Solve.

a) $x^4 - 81 = 0$

b) $x^3 - x^2 - 10x - 8 = 0$

c) $8x^3 + 27 = 0$

d) $12x^4 - 7x^2 - 6x + 16x^3 = 0$

15. A family of quartic functions has zeros −3, −1, and 1 (order 2).

a) Write an equation for the family. State two other members of the family.

b) Determine an equation for the member of the family that passes through the point (−2, −6).

c) Sketch the function you found in part b).

d) Determine the intervals where the function in part b) is positive.

16. Solve each inequality, showing the appropriate steps. Illustrate your solution on a number line.

a) $(x - 4)(x + 3) > 0$

b) $2x^2 + x - 6 < 0$

c) $x^3 - 2x^2 - 13x \leq 10$

Chapter 3 Rational Functions

17. Determine equations for the vertical and horizontal asymptotes of each function.

a) $f(x) = \dfrac{1}{x - 2}$

b) $g(x) = \dfrac{x + 5}{x + 3}$

c) $h(x) = \dfrac{2}{x^2 - 9}$

d) $k(x) = \dfrac{-1}{x^2 + 4}$

18. For each function,

i) determine equations for the asymptotes

ii) determine the intercepts

iii) sketch a graph

iv) describe the increasing intervals and the decreasing intervals

v) state the domain and the range

a) $f(x) = \dfrac{1}{x + 4}$ b) $g(x) = \dfrac{-4}{x - 2}$

c) $h(x) = \dfrac{x - 1}{x + 3}$ d) $i(x) = \dfrac{2x + 3}{5x + 1}$

e) $j(x) = \dfrac{10}{x^2}$ f) $k(x) = \dfrac{3}{x^2 - 6x - 27}$

19. Analyse the slope and the change in slope for the appropriate intervals of the function $f(x) = \dfrac{1}{x^2 - 4x - 21}$. Sketch a graph of the function.

20. Solve algebraically.

a) $\dfrac{5}{x - 3} = 4$

b) $\dfrac{2}{x - 1} = \dfrac{4}{x + 5}$

c) $\dfrac{6}{x^2 + 4x + 7} = 2$

21. Solve each inequality. Illustrate the solution on a number line.

a) $\dfrac{3}{x - 4} < 5$

b) $\dfrac{x^2 - 8x + 15}{x^2 + 5x + 4} \geq 0$

22. A lab technician pours a quantity of a chemical into a beaker of water. The rate, R, in grams per second, at which the chemical dissolves can be modelled by the function $R(t) = \dfrac{2t}{t^2 + 4t}$, where t is the time, in seconds.

a) By hand or using technology, sketch a graph of this relation.

b) What is the equation of the horizontal asymptote? What is its significance?

c) State an appropriate domain for this relation if a rate of 0.05 g/s or less is considered to be inconsequential.

Chapter 4 Trigonometry

23. Determine the exact radian measure for each angle.

a) $135°$ **b)** $-60°$

24. Determine the exact degree measure for each angle.

a) $\frac{\pi}{6}$ **b)** $\frac{9\pi}{8}$

25. A sector angle of a circle with radius 9 cm measures $\frac{5\pi}{12}$. What is the perimeter of the sector?

26. Determine the exact value of each trigonometric ratio.

a) $\cos\frac{5\pi}{6}$ **b)** $\sin\frac{3\pi}{2}$

c) $\tan\frac{4\pi}{3}$ **d)** $\cot\frac{11\pi}{4}$

27. Use the sum or difference formulas to find the exact value of each.

a) $\cos\frac{\pi}{12}$ **b)** $\sin\frac{11\pi}{12}$

28. Prove each identity.

a) $\sec x - \tan x = \frac{1 - \sin x}{\cos x}$

b) $(\csc x - \cot x)^2 = \frac{1 - \cos x}{1 + \cos x}$

c) $\sin 2A = \frac{2\tan A}{\sec^2 A}$

d) $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$

29. Given $\sin x = \frac{1}{5}$ and $\sin y = \frac{5}{6}$, where x and y are acute angles, determine the exact value of $\sin(x + y)$.

30. Given that $\cos\frac{5\pi}{8} = \sin y$, first express $\frac{5\pi}{8}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle y.

Chapter 5 Trigonometric Functions

31. a) State the period, amplitude, phase shift, and vertical translation for the function $f(x) = 3\sin\left[2\left(x - \frac{\pi}{2}\right)\right] + 4$.

b) State the domain and the range of $f(x)$.

32. Sketch a graph of each function for one period. Label the x-intercepts and any asymptotes.

a) $f(x) = \sin(x - \pi) - 1$

b) $f(x) = -3\cos\left[4\left(x + \frac{\pi}{2}\right)\right]$

c) $f(x) = \sec\left(x - \frac{\pi}{2}\right)$

33. Solve for $\theta \in [0, 2\pi]$.

a) $2\sin\theta = -\sqrt{3}$

b) $2\sin\theta\cos\theta - \cos\theta = 0$

c) $\csc^2\theta = 2 + \csc\theta$

34. The blade of a sabre saw moves up and down. Its vertical displacement in the first cycle is shown in the table.

Time (s)	Displacement (cm)
0	0
0.005	0.64
0.01	1.08
0.015	1.19
0.02	0.92
0.025	0.37
0.03	−0.30
0.035	−0.87
0.04	−1.18
0.045	−1.12
0.05	−0.71
0.055	−0.08
0.06	0.58
0.065	1.05
0.07	1.19

a) Make a scatter plot of the data.

b) Write a sine function to model the data.

c) Graph the sine function on the same set of axes as in part a).

d) Estimate the rate of change when the displacement is 0 cm, to one decimal place.

Chapter 6 Exponential and Logarithmic Functions

35. Express in logarithmic form.

a) $7^2 = 49$ **b)** $a^b = c$

c) $8^3 = 512$ **d)** $11^x = y$

36. a) Sketch graphs of $f(x) = \log x$ and $g(x) = \frac{1}{2}\log(x + 1)$ on the same set of axes. Label the intercepts and any asymptotes.

b) State the domain and the range of each function.

37. Express in exponential form.

a) $\log_3 6561 = 8$
b) $\log_a 75 = b$
c) $\log_7 2401 = 4$
d) $\log_a 19 = b$

38. Evaluate.

a) $\log_2 256$
b) $\log_{15} 15$
c) $\log_6 \sqrt{6}$
d) $\log_3 243$
e) $\log_{12} 12$
f) $\log_{11} \frac{1}{\sqrt{121}}$

39. Solve for x.

a) $\log_3 x = 4$
b) $\log_x 125 = 3$
c) $\log_7 x = 5$
d) $\log_x 729 = 6$
e) $\log_{\frac{1}{2}} 128 = x$
f) $\log_{\frac{1}{4}} 64 = x$

40. A culture begins with 100 000 bacteria and grows to 125 000 bacteria after 20 min. What is the doubling period, to the nearest minute?

41. The pH scale is defined as $\text{pH} = -\log[H^+]$, where $[H^+]$ is the concentration of hydronium ions, in moles per litre.

a) Eggs have a pH of 7.8. Are eggs acidic or alkaline? What is the concentration of hydronium ions in eggs?

b) A weak vinegar solution has a hydronium ion concentration of 7.9×10^{-4} mol/L. What is the pH of the solution?

Chapter 7 Tools and Strategies for Solving Exponential and Logarithmic Equations

42. Solve each equation. Check for extraneous roots.

a) $3^{2x} + 3^x - 21 = 0$

b) $4^x + 15(4)^{-x} = 8$

43. Use the laws of logarithms to evaluate.

a) $\log_8 4 + \log_8 128$
b) $\log_7 7\sqrt{7}$
c) $\log_5 10 - \log_5 250$
d) $\log_6 \sqrt[3]{6}$

44. Solve, correct to four decimal places.

a) $2^x = 13$
b) $5^{2x+1} = 97$
c) $3^x = 19$
d) $4^{3x+2} = 18$

45. Solve. Check for extraneous roots.

a) $\log_5(x + 2) + \log_5(2x - 1) = 2$

b) $\log_4(x + 3) + \log_4(x + 4) = \frac{1}{2}$

46. Determine the point(s) of intersection of the functions $f(x) = \log x$ and $g(x) = \frac{1}{2}\log(x + 1)$.

47. Bismuth is used in making chemical alloys, medicine, and transistors. A 10-mg sample of bismuth-214 decays to 9 mg in 3 min.

a) Determine the half-life of bismuth-214.

b) How much bismuth-214 remains after 10 min?

c) Graph the amount of bismuth-214 remaining as a function of time.

d) Describe how the graph would change if the half-life were shorter. Give reasons for your answer.

48. The volume of computer parts in landfill sites is growing exponentially. In 2001, a particular landfill site had 124 000 m^3 of computer parts, and in 2007, it had 347 000 m^3 of parts.

a) What is the doubling time of the volume of computer parts in this landfill site?

b) What is the expected volume of computer parts in this landfill site in 2020?

49. The value of a particular model of car depreciates by 18% per year. This model of car sells for $35 000.

a) Write an equation to relate the value of the car to the time, in years.

b) Determine the value of the car after 5 years.

c) How long will it take for the car to depreciate to half its original value?

d) Sketch a graph of this relation.

e) Describe how the shape of the graph would change if the rate of depreciation changed to 25%.

Chapter 8 Combining Functions

50. Consider $f(x) = 2^{-\frac{x}{\pi}}$ and $g(x) = 2\cos(4x)$ for $x \in [0, 4\pi]$. Sketch a graph of each function.

a) $y = f(x) + g(x)$ **b)** $y = f(x) - g(x)$

c) $y = f(x)g(x)$ **d)** $y = \frac{f(x)}{g(x)}$

51. Given $f(x) = 2x^2 + 3x - 5$ and $g(x) = x + 3$, determine each of the following.

a) $f(g(x))$ **b)** $g(f(x))$

c) $f(g(-3))$ **d)** $g(f(7))$

52. If $f(x) = \frac{1}{x}$ and $g(x) = 4 - x$, determine each of the following, if it exists.

a) $f(g(3))$ **b)** $f(g(0))$

c) $f(g(4))$ **d)** $g(f(4))$

53. Find expressions for $f(g(x))$ and $g(f(x))$, and state their domains.

a) $f(x) = \sqrt{x}$, $g(x) = x + 1$

b) $f(x) = \sin x$, $g(x) = x^2$

c) $f(x) = |x|$, $g(x) = x^2 - 6$

d) $f(x) = 2^{x+1}$, $g(x) = 3x + 2$

e) $f(x) = (x + 3)^2$, $g(x) = \sqrt{x - 3}$

f) $f(x) = \log x$, $g(x) = 3^{x+1}$

54. Consider $f(x) = -\frac{2}{x}$ and $g(x) = \sqrt{x}$.

a) Determine $f(g(x))$.

b) State the domain of $f(g(x))$.

c) Determine whether $f(g(x))$ is even, odd, or neither.

55. Verify, algebraically, that $f(f^{-1}(x)) = x$ for each of the following.

a) $f(x) = x^2 - 4$

b) $f(x) = \sin x$

c) $f(x) = 3x$

d) $f(x) = \frac{1}{x - 2}$

56. Solve. Illustrate each inequality graphically.

a) $\sin x < 0.1x^2 - 1$

b) $x + 2 \geq 2^x$

57. A Ferris wheel rotates such that the angle of rotation, θ, is defined by $\theta = \frac{\pi t}{15}$, where t is the time, in seconds. A rider's height, h, in metres, above the ground can be modelled by the function $h(\theta) = 20\sin\theta + 22$.

a) Write an equation for the rider's height in terms of time.

b) Sketch graphs of the three functions, on separate sets of axes, one above the other.

c) Compare the periods of the graphs of $h(\theta)$ and $h(t)$.

58. An office chair manufacturer models its weekly production since 2001 by the function $N(t) = 100 + 25t$, where t is the time, in years, since 2001, and N is the number of chairs. The size of the manufacturer's workforce can be modelled by the function $W(N) = 3\sqrt{N}$.

a) Write the size of the workforce as a function of time.

b) State the domain and range of the function in part a) that is relevant to this problem. Sketch its graph.

59. An environmental scientist measures the pollutant in a lake. The concentration, $C(P)$, in parts per million (ppm), of pollutant can be modelled as a function of the population, P, of the lakeside city, by $C(P) = 1.28P + 53.12$. The city's population, in ten thousands, can be modelled by the function $P(t) = 12.5 \times 2^{\frac{t}{20}}$, where t is the time, in years.

a) Determine an equation for the concentration of pollutant as a function of time.

b) Sketch a graph of this function.

c) How long will it take for the concentration to reach 100 ppm?